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Lowering operators associated with D-Laguerre–Hahn polynomials

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In this paper, a new lowering operator \(D_u\) with a linear functional \(u\) as a parameter is introduced in the linear space \(\mathbb{P}\) of all polynomials in one variable with complex coefficients. It is given by

\[
D_u(p)(x) = p'(x) + \left\langle u, \frac{p(x) - p(y)}{x - y} \right\rangle, \quad p \in \mathbb{P}.
\]

The concept of \(D_u\)-semiclassical polynomial sequence is defined. We show that such a sequence belongs to the family of \(D_u\)-Laguerre–Hahn polynomials. This allows us to provide some characterizations of a \(D_u\)-semiclassical polynomial sequence in terms of a distributional equation that the linear functional satisfies as well as a structure relation. An illustrative example is considered.

Keywords: linear functionals; quasi-orthogonality; orthogonal polynomials; Laguerre–Hahn polynomials; Appell polynomials

AMS Subject Classifications: 33C45; 42C05

1. Introduction

Let \(\mathbb{P}\) be the linear space of polynomials with complex coefficients. Let \(\mathbb{P}'\) be its dual space. \((u)_n := \langle u, x^n \rangle, \ n \geq 0,\) will denote the moments of \(u \in \mathbb{P}'\) with respect to the sequence \(\{x^n\}_{n \geq 0}.\) In the sequel, we set \(\mathbb{P}'_M := \{u \in \mathbb{P}', \ (u)_0 \neq -n, \ n \geq 1\}.\) When \((u)_0 = 1,\) the linear functional \(u\) is said to be normalized. Let us define the following operations on \(\mathbb{P}',\) (see [6,12]). For any \(c \in \mathbb{C},\) \(p, q \in \mathbb{P}\) and \(u, v \in \mathbb{P}',\) we have

\[
\langle qu, p \rangle = \langle u, qp \rangle, \quad \langle u', p \rangle = -\langle u, p' \rangle, \\
\langle \delta_c, p \rangle = p(c), \quad (\text{Dirac delta at point } c \in \mathbb{C}), \quad \delta := \delta_0,
\]

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\[ \langle uv, p \rangle = \langle v, up \rangle, \quad \text{where} \ (up)(x) = \left( u, \frac{xp(x) - yp(y)}{x - y} \right), \]

\[ \langle (x - c)^{-1}u, p \rangle = \langle u, \theta_c(p) \rangle = \left( u, \frac{p(x) - p(c)}{x - c} \right). \]

Here, \( \langle u, y, p(x, y) \rangle \) means the action of \( u \) on the polynomial \( p(x, y) \) with respect to the variable \( y \). We also denote \( \mathbb{C}^* = \mathbb{C}\setminus\{0\} \).

The following lemma will be useful in the sequel.

**Lemma 1.1** ([6]) For any \( u, v \in \mathbb{P}' \), \( f, g \in \mathbb{P} \) and \( c \in \mathbb{C} \), we have

\[
(x - c)((x - c)^{-1}u) = u, \\
(fu)' = fu' + f'u, \\
f(x^{-1}u) = x^{-1}(fu) + \langle u, \theta_0 f \rangle \delta, \\
f(uv) = (fu)v + x(u\theta_0 f)(x)v, \\
\langle u\theta_0(fg) \rangle(x) = g(x)(u\theta_0 f)(x) + (fu)\theta_0 g(x), \\
\langle u^2, \theta_0(fg) \rangle = \langle u, f(u\theta_0 g) + g(u\theta_0 f) \rangle. \]

A non-zero \( v \in \mathbb{P}' \) is said to be weakly regular if for a \( \phi \in \mathbb{P} \) such that \( \phi v = 0 \), then \( \phi = 0 \) [10]. Clearly, \( \delta_c \) is not a weakly regular functional for every \( c \in \mathbb{C} \), since \( (x - c)\delta_c = 0 \).

Let \( \{B_n\}_{n\geq0} \) be a monic polynomial sequence (MPS), \( \deg B_n = n \), and let \( \{w_n\}_{n\geq0} \) be its dual sequence defined by \( \langle w_n, B_m \rangle = \delta_{n,m}, \ n, \ m \geq 0 \), where \( \delta_{n,m} \) is the Kronecker delta. The MPS \( \{B_n\}_{n\geq0} \) is said to be orthogonal (MOPS) with respect to \( w \in \mathbb{P}' \), if \( \langle w, B_n B_m \rangle = 0, \ n \neq m \) and \( \langle w, B_n^2 \rangle \neq 0, \ n \geq 0 \). In this case, \( w \) is said to be quasi-definite (regular) [5]. Note that, \( w = (w)_{0}w_{0} \), with \( (w)_{0} \neq 0 \). Referring to [10], quasi-definite linear functionals are weakly regular. Note that, in general, the converse is not true.

**Proposition 1.2** ([12]) An MPS \( \{B_n\}_{n\geq0} \) with dual sequence \( \{w_n\}_{n\geq0} \) is orthogonal with respect to \( w_{0} \) if and only if one of the following statements hold.

(i) \( w_{n} = \langle w_{0}, B_{n}^2 \rangle^{-1}B_{n}w_{0}, \ n \geq 0 \).

(ii) \( \{B_{n}\}_{n\geq0} \) satisfies a three-term recurrence relation (TTRR).

\[
B_{0}(x) = 1, \quad B_{1}(x) = x - \beta_{0}, \\
B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_{n}(x), \ n \geq 0, \]

where \( \beta_{n} = \langle w_{0}, xB_{n}^2 \rangle / \langle w_{0}, B_{n}^2 \rangle \in \mathbb{C} \) and \( \gamma_{n+1} = \langle w_{0}, B_{n+1}^2 \rangle / \langle w_{0}, B_{n}^2 \rangle \in \mathbb{C}^*, \ n \geq 0 \).

Recall that \( w \in \mathbb{P}' \) is quasi-definite if and only if for each non-negative integer \( n \) the Hankel determinant \( \Delta_{n}(w) = \det((w)_{i+j})_{0\leq i,j \leq n} \neq 0, \ n \geq 0 \) [5].

For an MOPS \( \{B_{n}\}_{n\geq0} \), we can define the sequence of associated polynomials of the first kind \( \{B_{n}^{(1)}\}_{n\geq0} \) as follows [5]: \( B_{n}^{(1)}(x) = w_{0}\theta_{0}B_{n+1}(x), \ n \geq 0 \). The sequence \( \{B_{n}^{(1)}\}_{n\geq0} \) is also a MOPS.
and satisfies the shifted TTRR

\[ B_{0}^{(1)}(x) = 1, \quad B_{1}^{(1)}(x) = x - \beta_{1}, \]

\[ B_{n+2}^{(1)}(x) = (x - \beta_{n+2})B_{n+1}^{(1)}(x) - \gamma_{n+2}B_{n}^{(1)}(x), \quad n \geq 0. \]

An MOPS \( \{B_n\}_{n \geq 0} \) with respect to \( w_0 \) is said to be of D-Laguerre–Hahn if \( w_0 \) satisfies the following distributional equation (see [3,4,8]):

\[ (\Phi w_0)' + (\Psi_1 w_0) + B(x^{-1}w_0^2) = 0, \quad (1.8) \]

where \( (\Phi, \Psi_1, B) \in \mathbb{P}^3 \) and \( \Phi \) is a monic polynomial. In this case, \( w_0 \) is said to be a D-Laguerre–Hahn linear functional. We can associate with \( (\Phi, \Psi_1, B) \) the non-negative integer number: \( \max(\deg(\Phi) - 2, \deg(B) - 2, \deg(\Psi_1) - 1) \). Obviously, a D-Laguerre–Hahn linear functional \( w_0 \) satisfies an infinite number of equations like (1.8). Indeed, it is enough to multiply by any \( \chi \in \mathbb{P} \) on both sides of (1.8) in such a way that \( \chi(\Phi w_0)' + (\chi \Psi_1 - \chi'\Phi)w_0 + (\chi B)(x^{-1}w_0^2) = 0 \). So, we can associate with \( w_0 \) a non-negative integer \( s \), the so-called class of \( w_0 \), which is defined by \( s := \min(\max(\deg(\Phi) - 2, \deg(B) - 2, \deg(\Psi_1) - 1)) \), where the minimum is taken over all possible choices of polynomials \( (\Phi, \Psi_1, B) \) satisfying (1.8). It is well known (see [1]) that a D-Laguerre–Hahn linear functional \( w_0 \) satisfying (1.8) is of class \( s \), if and only if for any zero \( c \) of \( \Phi \), we have \( |\Psi_1(c) + \Phi'(c)| + |B(c)| + |[w_0, \theta, \Psi_1 + \theta^2\Phi + w_0(\theta_0 B)]| > 0 \).

When \( B \) vanishes, \( \{B_n\}_{n \geq 0} \) or \( w_0 \) is said to be D-semiclassical. In particular, a D-semiclassical polynomial sequence (respectively, a linear functional) of class zero is said to be D-classical. By means of a suitable affine transformation, there exist four canonical classical cases, the well-known families of Hermite, Laguerre, Bessel and Jacobi MOPS, respectively (see [2,3,10]).

If \( B \neq 0 \), then \( \{B_n\}_{n \geq 0} \) or \( w_0 \) is said to be of strict D-Laguerre–Hahn. In the same way, by means of a suitable affine transformation, there exist four families of D-Laguerre–Hahn polynomials of class zero. Each of them includes two subfamilies: a singular family and a non-singular one. For more details, see [4].

**Proposition 1.3** ([7]) Let \( \{B_n\}_{n \geq 0} \) be an MOPS with respect to \( w_0 \), satisfying (1.7). The following statements are equivalent.

(i) \( w_0 \) is a D-Laguerre–Hahn linear functional of class \( s \) satisfying (1.8).

(ii) \( \{B_n\}_{n \geq 0} \) satisfies a first structure relation (FSR)

\[ \Phi(x)B_{n+1}'(x) - B(x)B_n^{(1)}(x) = \frac{C_{n+1}(x) - C_0(x)}{2}B_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)B_n(x), \quad n \geq 0, \]

(1.9)

where \( C_n \) and \( D_n \) are polynomials with coefficients depending on \( n \), such that \( \deg C_n \leq s + 1, \deg D_n \leq s \), and given by

\[ C_{n+1}(x) = -C_n(x) + 2(x - \beta_n)D_n(x), \]

\[ \gamma_{n+1}D_{n+1}(x) = -\Phi(x) + \gamma_nD_{n-1}(x) - (x - \beta_n)C_n(x) + (x - \beta_n)^2D_n(x), \]

for every \( n \geq 0 \), and the initial conditions

\[ C_0(x) = -\Phi'(x) - \Psi_1(x), \]

\[ D_0(x) = -(w_0\theta_0\Phi)'(x) - w_0\theta_0(\Psi_1)(x) - (w_0^2\theta_0^2B)(x), \quad D_{-1}(x) = 0. \]
Using the orthogonality of \( \{B_n\}_{n \geq 0} \), (1.9) can be written as follows:

\[
\Phi(x)B'_{n+1}(x) - B(x)B_{n+1}^{(1)}(x) = \sum_{v=n-s}^{n+d} \lambda_{n,v}B_v(x), \quad n \geq s, \tag{1.10}
\]

where \( d = \max(\deg \Phi, \deg B) \) and \( s \) is the class of the linear functional \( u_0 \).

In this paper, we are dealing with the non-singular \( D \)-Laguerre–Hahn MOPS of class zero analogous to the Hermite MOPS that we denote by \( \{\hat{H}_n(., \xi)\}_{n \geq 0} \), where \( \xi = (\tau, \lambda, \rho) \in \mathbb{C}^3 \), with \( \tau \neq -n, \ n \geq 1, \) and \( \rho \neq 0 \). Such a polynomial sequence is orthogonal with respect to a unique monic linear functional \( \mathcal{H}(\xi) \). Recall that \( \{\hat{H}_n(., \xi)\}_{n \geq 0} \) appears in the description of the Laguerre–Hahn MOPS of class zero [4]. In the sequel, the following properties of \( \{\hat{H}_n(., \xi)\}_{n \geq 0} \) will be needed.

\[
\begin{align*}
\text{(TTRR)} & & \hat{H}_0(x; \xi) = 1, & & \hat{H}_1(x; \xi) = x - \lambda, & & \hat{H}_{n+2}(x; \xi) = x\hat{H}_{n+1}(x; \xi) - \gamma_{n+1}\hat{H}_n(x; \xi), & & n \geq 0, \\
\text{(FSR)} & & \hat{H}'_{n+1}(x; \xi) - B(x)\hat{H}_{n+1}^{(1)}(x; \xi) = -2\rho^{-1}(\rho - 1)x + \lambda)\hat{H}_{n+1}(x; \xi), & & (n + \tau + 1)\hat{H}_n(x; \xi) + (\tau + 1)(\rho - 1)\delta_{n,0}, & & n \geq 0, 
\end{align*}
\tag{1.11}
\]

where \( \gamma_1 = (\rho/2)(\tau + 1), \ \gamma_{n+1} = (\frac{1}{2})(n + \tau + 1), \ n \geq 1. \)

where \( B(x) = 2\rho^{-1}(\rho - 1)x^2 + 2\rho^{-1}\lambda(2 - \rho)x + 1 - \rho(\tau + 1) - 2\rho^{-1}\lambda^2 \).

Note that \( \{\hat{H}_n(., \xi)\}_{n \geq 0} \) is a generalization of the Hermite MOPS \( \{\hat{H}_n\}_{n \geq 0} \), which appears when \( \xi = (0, 0, 1, 0, 1) \). It is well known that \( \{\hat{H}_n\}_{n \geq 0} \) is an Appell sequence with respect to the usual derivative operator \( D = \frac{d}{dx} \) or, simply, a \( D \)-AMPS, since \( D(\hat{H}_{n+1}) = (n + 1)\hat{H}_n, \ n \geq 0 \) [9]. Let us show that \( \{\hat{H}_n(., \xi)\}_{n \geq 0} \) satisfies a similar property with respect to a new operator. Indeed, from the expression of \( B(x) \) already quoted and (1.11), we obtain \( (B\mathcal{H}(\xi))\theta_0\hat{H}_{n+1}(x; \xi) = -2\rho^{-1}((\rho - 1)x + \lambda)\hat{H}_{n+1}(x; \xi) + B(x)\hat{H}_{n+1}^{(1)}(x; \xi) + (\tau + 1)(\rho - 1)\delta_{n,0}, n \geq 0. \) Accordingly, \( \hat{H}'_{n+1}(x; \xi) - (B\mathcal{H}(\xi))\theta_0\hat{H}_{n+1}(x; \xi) = (n + \tau + 1)\hat{H}_n(x; \xi), n \geq 0, \) by (1.12). If we set \( \mathcal{U}(\xi) := -B\mathcal{H}(\xi) \in \mathbb{P}' \), then \( \mathcal{U}(\xi)_0 = -\tau \neq -n, n \geq 1, \) and we can write \( \hat{H}'_{n+1}(x; \xi) + \mathcal{U}(\xi)\theta_0\hat{H}_{n+1}(x; \xi) = (n + \tau + 1)\hat{H}_n(x; \xi), n \geq 0. \) Hence, by introducing the operator \( D_{\mathcal{U}(\xi)}(p)(x) := p'(x) + \mathcal{U}(\xi)\theta_0p(x) \), for all \( p \in \mathbb{P} \), it follows that

\[
D_{\mathcal{U}(\xi)}\hat{H}_{n+1}(x; \xi) = (n + \tau + 1)\hat{H}_n(x; \xi), \quad n \geq 0. \tag{1.13}
\]

Thus, \( \{\hat{H}_n(., \xi)\}_{n \geq 0} \) is a \( D_{\mathcal{U}(\xi)} \)-AMPS. Besides, from (1.11), it follows that

\[
D_{\mathcal{U}(\xi)}^2\hat{H}_{n+1}(x; \xi) - 2xD_{\mathcal{U}(\xi)}\hat{H}_{n+1}(x; \xi) = -2(n + \tau + 1)\hat{H}_{n+1}(x; \xi), \quad n \geq 2.
\]

From a more general point of view, for a given \( u \in \mathbb{P}_M \), let \( D_u : \mathbb{P} \rightarrow \mathbb{P} \) be the linear operator defined by

\[
D_u(p)(x) = p'(x) + \left( u_y, \frac{p(x) - p(y)}{x - y} \right), \quad p \in \mathbb{P}, \quad (D_0 = D).
\]

The aim of this work is to introduce the concept of \( D_u \)-semiclassical MOPS and to show that they belong to the \( D \)-Laguerre–Hahn class. A necessary and sufficient condition for a \( D \)-Laguerre–Hahn MOPS to be a \( D_u \)-semiclassical MOPS will be given.
The structure of the paper is as follows. Section 2 contains some properties of the operator $D_u$ as well as the definition of the $D_u$-semiclassical polynomials. In Section 3, we give a characterization of a $D_u$-semiclassical linear functional by a linear distributional equation. In Section 4, first we characterize a $D_u$-semiclassical MOPS by means of an FSR, i.e. a finite-type relation between the polynomials and the monic polynomials of its first $D_u$-derivative [11]. Second, we will prove that every $D_u$-semiclassical linear functional belongs to the family of $D$-Laguerre–Hahn linear functionals. Finally, an example of $D_u$-semiclassical MOPS is presented.

2. The $D_u$-semiclassical polynomials

2.1. Some properties of the operator $D_u$

Among the properties of the linear operator $D_u$, we emphasize the following one related to the lowering character of such operator. Indeed, we have

$$D_u(1) = 0, \quad D_u(x) = (u)_0 + 1, \quad D_u(x^n) = (n + (u)_0)x^{n-1} + \sum_{\nu=0}^{n-2} (u)_{n-\nu-1} x^\nu, \quad n \geq 2.$$  

The assumption $u \in \mathbb{P}_M^\prime$ is related to the fact that $\text{deg}(D_u(p)) = \text{deg}(p) - 1$, for every polynomial $p$.

The transpose $D_u$ of $D_u$ is defined as follows. For all $(p, w) \in \mathbb{P} \times \mathbb{P}^\prime$, \(\langle D_u(w), p \rangle = \langle w, D_u(p) \rangle = \langle w, p' + u\theta_0 p \rangle = \langle -w' + x^{-1}(uw), p \rangle\). Thus, \(\langle D_u(w), p \rangle = -w' + x^{-1}(uw), \quad w \in \mathbb{P}^\prime\). For the sake of simplicity, we will denote $D_u := -D_u$, and then we have: $\langle D_u(w), p \rangle = -\langle w, D_u(p) \rangle, \quad (p, w) \in \mathbb{P} \times \mathbb{P}^\prime$.

The linear operator $D_u$ is one-to-one. Indeed, if $D_u(w) = 0$, $w \in \mathbb{P}^\prime$, then $\langle w, D_u(x^{n+1}) \rangle = 0, \quad n \geq 0$. Hence, $w = 0$, since we have $\text{deg}(D_u(x^{n+1})) = n$.

The proof of the following operational rules is a straightforward exercise.

**Proposition 2.1** For $v \in \mathbb{P}^\prime$ and $(f, g) \in \mathbb{P}^2$, we have

$$D_u(f g) = D_u(f)g + fD_u(g) + u\theta_0(fg) - (u\theta_0 f)g - f(u\theta_0 g), \quad (2.1)$$  

$$D_u(f v) = fD_u(v) + D_u(f)v + (v\theta_0 f)u - (u\theta_0 f)v. \quad (2.2)$$  

For an MPS $\{B_n\}_{n \geq 0}$ and $u \in \mathbb{P}_M^\prime$, we can define

$$B_n^{[1]}(x; u) := (n + (u)_0 + 1)^{-1}(D_uB_{n+1})(x), \quad n \geq 0. \quad (2.3)$$  

Clearly, $\{B_n^{[1]}(\cdot; u)\}_{n \geq 0}$ is an MPS, $\text{deg} B_n^{[1]}(\cdot; u) = n, \quad n \geq 0$. If $\{w_n^{[1]}(u)\}_{n \geq 0}$ denotes the dual sequence of $\{B_n^{[1]}(\cdot; u)\}_{n \geq 0}$, then we have the following.

**Lemma 2.2**

$$D_u(w_n^{[1]}(u)) = -(n + (u)_0 + 1)w_{n+1}, \quad n \geq 0. \quad (2.4)$$  

**Proof** Let $n \geq 0$ be a fixed integer. Since $\langle w_n^{[1]}(u), B_m^{[1]}(\cdot; u) \rangle = \delta_{n,m}$, then $\langle D_u(w_n^{[1]}(u)), B_{m+1} \rangle = -(m + (u)_0 + 1)\delta_{n,m} = -(n + (u)_0 + 1)\delta_{n,m}, \quad m \geq 0$. In addition, $\langle D_u(w_n^{[1]}(u)), 1 \rangle = 0$. Hence, $D_u(w_n^{[1]}(u)) = -(n + (u)_0 + 1)w_{n+1}$. ■
2.2. The $\mathcal{D}_n$-semiclassical MOPS

We start by recalling the notion of the quasi-orthogonality and some of its useful results (see [5,13]).

**Definition 2.3** Let $v \in \mathbb{P}'$ and $\sigma$ is a non-negative integer. A MOPS $\{B_n\}_{n \geq 0}$ is said to be quasi-orthogonal of order $\sigma$ with respect to $v$, if

$$\langle v, x^m B_n(x) \rangle = 0, \quad 0 \leq m \leq n - \sigma - 1, \quad n \geq \sigma + 1, \quad (2.5)$$

$$\exists r \geq \sigma, \quad \langle v, x^{r-\sigma} B_r(x) \rangle \neq 0. \quad (2.6)$$

The MOPS $\{B_n\}_{n \geq 0}$ is said to be strictly quasi-orthogonal of order $\sigma$ with respect to $v$, if it satisfies (2.5), and

$$\langle v, x^{n-\sigma} B_n \rangle \neq 0, \quad n \geq \sigma. \quad (2.7)$$

**Remark 2.4**

(i) Here, $v$ is not assumed to be quasi-definite. In general, a quasi-orthogonal sequence can not be strictly quasi-orthogonal.

(ii) In particular, a strictly quasi-orthogonal sequence of order zero with respect to $v$ is an orthogonal sequence with respect to $v$.

In order to give a suitable definition of a $\mathcal{D}_n$-semiclassical MOPS, we need the following results.

**Lemma 2.5** Let $\{B_n\}_{n \geq 0}$ be a MOPS with respect to $w_0$. Then, for any integer $\sigma \geq 0$ there exists $A_{\sigma+1} \in \mathbb{P}$, $\deg A_{\sigma+1} = \sigma + 1$, such that

$$\langle w_0, A_{\sigma+1} \rangle = 0, \quad (2.8)$$

$$\langle w_0, x A_{\sigma+1} \rangle \neq -n, \quad n \geq 0. \quad (2.9)$$

**Proof** Assume that $\{B_n\}_{n \geq 0}$ is a MOPS with respect to $w_0$. So, $w_0$ is quasi-definite and hence $\Delta_1(w_0) \neq 0$, i.e. $(w_0)_2 - (w_0)_1^2 \neq 0$.

If $\sigma = 0$, we can take $A_1(x) = \lambda(x - (w_0)_1)$, with $\lambda \neq -n((w_0)_2 - (w_0)_1^2)^{-1}$, $n \geq 0$. We get $\langle w_0, A_1 \rangle = \lambda((w_0)_1 - (w_0)_1) = 0$ and $\langle w_0, x A_1 \rangle = \lambda((w_0)_2 - (w_0)_1^2) \neq -n$, $n \geq 0$.

If $\sigma \geq 1$, we can take $A_{\sigma+1}(x) = x^{\sigma+1} + \theta(x - (w_0)_1) - (w_0)_{\sigma+1}$, with $\theta \neq (-n - (w_0)_{\sigma+2} + (w_0)_{\sigma+1}(w_0)_1)((w_0)_2 - (w_0)_1^2)^{-1}$, $n \geq 0$. Then $\langle w_0, A_{\sigma+1} \rangle = 0$, and $\langle w_0, x A_{\sigma+1} \rangle = (w_0)_{\sigma+2} + \theta((w_0)_2 - (w_0)_1^2) - (w_0)_{\sigma+1}(w_0)_1 \neq -n$, $n \geq 0$. \hfill \blacksquare

**Proposition 2.6** For any MOPS $\{B_n\}_{n \geq 0}$ with respect to $w_0$ and any integer $\sigma \geq 0$, there exists $A_{\sigma+1} \in \mathbb{P}$, $\deg A_{\sigma+1} = \sigma + 1$, such that for any $c \in \mathbb{C}$, the MOPS $\{B^{[i]}_{\sigma}(c; u)\}_{n \geq 0}$, where $u = -\delta_c + (x - c)A_{\sigma+1}w_0$, is quasi-orthogonal of order $\sigma$ with respect to $\delta_c$.

**Proof** Assume that $\{B_n\}_{n \geq 0}$ is a MOPS with respect to $w_0$ and $\sigma$ a non-negative integer. From Lemma 2.5, there exists $A_{\sigma+1} \in \mathbb{P}$, $\deg A_{\sigma+1} = \sigma + 1$ such that (2.8) and (2.9) hold. Given $c \in \mathbb{C}$ and let $u = -\delta_c + (x - c)A_{\sigma+1}w_0 \in \mathbb{P}'$. Since, $(u)_0 = 0 = 1 + \langle w_0, x A_{\sigma+1} \rangle - c \langle w_0, A_{\sigma+1} \rangle = -n + \langle w_0, x A_{\sigma+1} \rangle \neq -n$, $n \geq 1$, then $\{B^{[i]}_{\sigma}(c; u)\}_{n \geq 0}$ is an MOPS, $\deg B^{[i]}_{\sigma}(c; u) = n$, $n \geq 0$.

Next, let us show that $B^{[i]}_{\sigma}(c; u) = 0$, $n \geq \sigma + 1$, and $B^{[i]}_{\sigma}(c; u) \neq 0$, i.e. $\langle \delta_c, x^m \rangle x^{m} B^{[i]}_{\sigma}(c; u) = 0$, $0 \leq m \leq n - \sigma - 1$, $n \geq \sigma + 1$, and $\langle \delta_c, B^{[i]}_{\sigma}(c; u) \rangle \neq 0$, and then $\{B^{[i]}_{\sigma}(c; u)\}_{n \geq 0}$ is quasi-orthogonal of order $\sigma$ with respect to $\delta_c$.

Indeed, by (2.3), $B^{[i]}_{\sigma}(c; u) = (n + (u)_0 + 1)^{-1}(\delta'_c - x^{-1}(u \delta_c), B_{n+1})$. Since $\delta'_c = -x \delta'_c$ and $(x - c)A_{\sigma+1}w_0)\delta_c = xA_{\sigma+1}w_0$, then $u \delta_c = x(\delta'_c + A_{\sigma+1}w_0)$, and hence $x^{-1}(u \delta_c) = \delta'_c + ...
Applying $A_{\sigma+1} w_0$. So, $B_n^{[1]}(c; u) = -(n + (u)_0 + 1)^{-1}(w_0, A_{\sigma+1} B_{n+1})$, $n \geq 0$. Using the orthogonality of \( \{B_n\}_{n \geq 0} \), we get $B_n^{[1]}(c; u) = 0$, $n \geq \sigma + 1$, and $B^{[1]}_\sigma(c; u) = -K(w_0, B^{[2]}_\sigma)/(\sigma + (u)_0 + 1) \neq 0$, where $K$ is the leading coefficient of $A_{\sigma+1}$.

To avoid analogous situations to those found in the above proposition, we give the following definition of a $D_u$-semiclassical MOPS.

**Definition 2.7** Let $u \in \mathbb{R}_+$. An MPS $\{B_n\}_{n \geq 0}$ is said to be $D_u$-semiclassical, when it is orthogonal and such that the MPS $\{B_n^{[1]}(:, u)\}_{n \geq 0}$ given by (2.3) is quasi-orthogonal of order $\sigma$ with respect to a weak-regular linear functional $v$.

The quasi-definite linear functional $w_0$ corresponding to $\{B_n\}_{n \geq 0}$ is also said to be $D_u$-semiclassical.

As a straightforward consequence, the non-singular $D$-Laguerre–Hahn MOPS of class zero analogous to the Hermite polynomial sequence, is $D_{u(\xi)}$-semiclassical, according to (1.13) and Remark 2.4(ii).

3. Characterization of a $D_u$-semiclassical linear functional by means of a distributional equation

To give a characterization of a $D_u$-semiclassical linear functional by a distributional equation, we will need some information concerning $u$ and $v$.

**Lemma 3.1** Under the assumption of the previous definition, we have the following.

(i) There exists a non-zero $\phi \in \mathbb{P}$, $\deg \phi = t \leq \sigma + 2$, such that $v = \phi w_0$.

(ii) If $(v)_0 \neq 0$, then there exists $\hat{B} \in \mathbb{P}$, $\deg \hat{B} \leq \sigma + 2$, such that $u = \hat{B} w_0$.

**Proof** From the assumption, there exist a weak-regular linear functional $v$ and a non-negative integer $r \geq \sigma$, such that

\[
\langle v, x^m B^{[1]}_n(x; u) \rangle = 0, \quad 0 \leq m \leq n - \sigma - 1, \quad n \geq \sigma + 1, \tag{3.1}
\]

\[
\langle v, x^{r-\sigma} B^{[1]}_r(x; u) \rangle \neq 0. \tag{3.2}
\]

Applying $D_u$ on both sides of (1.7) and using (2.3) and (2.1), we obtain

\[
B_{n+1} + \langle u, B_{n+1} \rangle = (n + (u)_0 + 2)B^{[1]}_{n+1}(x; u) + (n + (u)_0 + 1)\beta_{n+1} B^{[1]}_n(x; u)
+ (n + (u)_0)\gamma_{n+1} B^{[1]}_{n-1}(x; u) - (n + (u)_0 + 1)x B^{[1]}_n(x; u), \quad n \geq 0. \tag{3.3}
\]

Then, $\langle v, x^m B_{n+1} \rangle + \langle u, B_{n+1} \rangle(v)_m = 0$, $n \geq m + \sigma + 2$, $m \geq 0$, due to (3.1) and (3.3). So, $\langle x^m v + (v)_m u, B_n \rangle = 0$, $n \geq \sigma + 3 + m$, $m \geq 0$. Using the orthogonality of $\{B_n\}_{n \geq 0}$, there exists a polynomial sequence $\{\Omega_m^{\sigma+2}(x)\}_{n \geq 0}$, $\deg \Omega_m^{\sigma+2} \leq m + \sigma + 2$, such that

\[
x^m v + (v)_m u = \Omega_m^{\sigma+2} w_0, \quad m \geq 0, \tag{3.4}
\]

where $\Omega_m^{\sigma+2}(x) = \sum_{\nu=0}^{m+\sigma+2} (x^\nu v + (v)_m u, B_\nu) (w_0, B^{[2]}_\nu)^{-1} B_\nu(x)$, $m \geq 0$. Integral Transforms and Special Functions
For $m = 0$, (3.4) gives
\[ v + (v)_0u = \Omega_{\sigma+2}w_0. \] (3.5)
Substituting (3.5) into (3.4), we obtain
\[ ((v)_m - (v)_0x^m)u = (\Omega_{m+\sigma+2} - x^m\Omega_{\sigma+2})w_0, \quad m \geq 0. \] (3.6)
So, we need to discuss two cases.

(A) $(v)_0 = 0$. Directly, (i) holds, with $\phi = \Omega_{\sigma+2}$, due to (3.5) with $(v)_0 = 0$, and the weak-regularity of $v$ to obtain $\deg \phi \geq 0$. In this case, (3.6) becomes
\[ (v)_mu = (\Omega_{m+\sigma+2} - x^m\Omega_{\sigma+2})w_0, \quad m \geq 0. \] (3.7)
From the weak-regularity of $v$, there exists an integer $k \geq 1$ such that
\[ k = \min\{m \geq 1 \mid (v)_m \neq 0\}. \] (3.8)
Thus, for $m = k$, (3.7) yields
\[ u = \hat{B}w_0, \] (3.9)
where $\hat{B}(x) = (v)_k^{-1}(\Omega_{k+\sigma+2}(x) - x^k\Omega_{\sigma+2}(x))$.

Besides, by (3.9), (3.7) and the quasi-definiteness of $w_0$, we obtain
\[ \Omega_{m+\sigma+2}(x) = x^m\Omega_{\sigma+2}(x) + (v)_m\hat{B}(x), \quad m \geq 0. \] (3.10)
Hence, if $(v)_0 = 0$ then (ii) holds.

(B) $(v)_0 \neq 0$. There exists an integer $l \geq 2$ such that $(v)_l \neq (v)_0^{-1}(v)_l^l$. Otherwise, if we set $c = (v)_1(v)_0^{-1}$, then $(v)_m = (v)_0c^m = 0$, $m \geq 0$, i.e. $v = (v)_0\delta_c$. This contradicts the weak-regularity of $v$.

By taking successively $m = 1$ and $m = l$ in (3.6), we obtain
\[ ((v)_1 - (v)_0x)u = (\Omega_{\sigma+3} - x\Omega_{\sigma+2})w_0, \] (3.11)
\[ ((v)_l - (v)_0x^l)u = (\Omega_{l+\sigma+2} - x^l\Omega_{\sigma+2})w_0. \] (3.12)
Since, $(v)_l \neq (v)_0^{-1}(v)_l^l$, then $(v)_l - (v)_0x^l$ and $(v)_1 - (v)_0x$ are coprimes. From Bézout’s identity, there exist two polynomials $E(x)$ and $F(x)$ such that
\[ ((v)_l - (v)_0x^l)E(x) + ((v)_1 - (v)_0x)F(x) = 1. \] (3.13)
By (3.11)–(3.13), we can easily deduce that
\[ u = \hat{B}w_0, \] (3.14)
where $\hat{B}(x) = E(x)(\Omega_{\sigma+3}(x) - x\Omega_{\sigma+2}(x)) + F(x)(\Omega_{l+\sigma+2}(x) - x^l\Omega_{\sigma+2}(x))$.

Hence, if $(v)_0 \neq 0$, then (ii) holds.

By inserting (3.14) in (3.6) and using the quasi-definiteness of $w_0$, we get \((v)_m - (v)_0x^m)\hat{B}(x) = \Omega_{m+\sigma+2}(x) - x^m\Omega_{\sigma+2}(x), m \geq 0\). The consideration of the degrees on both sides of the previous formula yields $\deg \hat{B} \leq \sigma + 2$. Substituting (3.14) into (3.5), we get $v = \phi w_0$ with $\phi(x) = \Omega_{\sigma+2}(x) - (v)_0\hat{B}(x)$, and $0 \leq \deg \phi \leq \sigma + 2$. Hence, if $(v)_0 \neq 0$, then (i) holds.

Next we will give a characterization of a $D_\sigma$-semiclassical MOPS.
THEOREM 3.2 Let $u \in \mathbb{P}_m^r$ and let $\{B_n\}_{n \geq 0}$ be an MOPS with respect to $w_0$. The following statements are equivalent.

(i) $\{B_n\}_{n \geq 0}$ is $D_u$-semiclassical.

(ii) There exists $(\Phi, \Psi, \hat{B}) \in \mathbb{P}_2$, $\Phi$ monic, $\deg \Phi = t$, $\deg \Psi = p \geq 1$, and $\deg \hat{B} \leq k + \sigma + 2$, where $k = \min\{m \geq 0 \mid (\Phi w_0)_m \neq 0\}$, and $\sigma = \max(t - 2, p - 1)$, such that $w_0$ is a solution of the distributional equation:

$$D_u(\Phi w_0) + \Psi w_0 = 0 \quad \text{where} \quad u = \hat{B} w_0. \quad (3.15)$$

Proof (i) $\Rightarrow$ (ii). From Lemma 3.1(i), let us write $\phi(x) = \lambda \Phi(x)$, where $\lambda$ is a normalization constant and $\Phi$ is a monic polynomial with $\deg \Phi = t \leq \sigma + 2$. From (2.3), we get

$$\langle D_u(\Phi w_0), B_{n+1} \rangle = -\lambda^{-1}(n + (u)0 + 1)\langle v, B_n^{[1]}(x; u) \rangle, \quad n \geq 0. \quad (3.16)$$

But, we have $\langle v, B_n^{[1]}(x; u) \rangle = 0$, $n \geq \sigma + 1$, according to (3.1), with $m = 0$. As a consequence, since $v \neq 0$, there exists an integer $p$, $1 \leq p \leq \sigma + 1$, such that $\langle v, B_p^{[1]}(x; u) \rangle \neq 0$. Using (3.16), it follows that $\langle D_u(\Phi w_0), B_p \rangle \neq 0$ and $\langle D_u(\Phi w_0), B_n \rangle = 0$, $n \geq p + 1$. So, from the orthogonality of $\{B_n\}_{n \geq 0}$, there exists a polynomial $\Psi(x) = \sum_{i=1}^{p} (v + (u)0)\lambda^{-1} \langle v, B_v^{[1]}(.; u) \rangle \langle w_0, B_v^{-1} \rangle B_v(x)$, $\deg \Psi = p \geq 1$, such that $D_u(\Phi w_0) + \Psi w_0 = 0$. Hence, (ii) holds.

Before showing that (ii) $\Rightarrow$ (i), we can determine the nature of the quasi-orthogonality, i.e. if it is strict or not.

For every integer $n \geq \sigma$, let us consider,

$$(n + (u)0 + 1)\langle v, x^{n-\sigma} B_n^{[1]}(x; u) \rangle = -\lambda \langle D_u(x^{n-\sigma} \Phi w_0), B_{n+1} \rangle. \quad (3.17)$$

But, from (2.2), we can write $D_u(x^{n-\sigma} \Phi w_0) = x^{n-\sigma} D_u(\Phi w_0) + \Phi D_u(x^{n-\sigma}) w_0 + \frac{1}{\Phi(\Phi w_0) \theta_0 x^{n-\sigma}} \hat{B} - \frac{(\hat{B} w_0) \theta_0 x^{n-\sigma}}{\Phi(\Phi w_0)} \Phi w_0$. So, in view of (3.15), we obtain

$$D_u(x^{n-\sigma} \Phi w_0) = \Lambda_{n+1} w_0, \quad n \geq \sigma \quad (3.18)$$

where

$$\Lambda_{n+1}(x) = -\lambda^{n-\sigma} \Psi(x) + \Phi(x) D_u(x^{n-\sigma})$$

$$+ \frac{1}{\Phi(\Phi w_0) \theta_0 x^{n-\sigma}} \hat{B}(x) - \frac{(\hat{B} w_0) \theta_0 x^{n-\sigma}}{\Phi(\Phi w_0)} \Phi(x). \quad (3.19)$$

Note that $\deg \Lambda_{n+1} \leq n + 1$, $n \geq \sigma$. Indeed, we will analyse the two previous situations.

(A) $\langle v \rangle_0 = 0$. By Lemma 3.1, $\deg \Phi = t \leq \sigma + 2$, $\deg \Psi = p \leq \sigma + 1$ and $\deg \hat{B} = \tilde{r} \leq k + \sigma + 2$, where $k$ is given by (3.8).

Note that $\deg((\Phi w_0) \theta_0 x^{n-\sigma}) = n - \sigma - k - 1$, because we have $v = \lambda \Phi w_0$, $\langle v \rangle_0 = \cdots = \langle v \rangle_{k-1} = 0$ and $\langle v \rangle_k \neq 0$. Since $\deg D_u(x^{n-\sigma}) = n - \sigma - 1$, and $\deg((\hat{B} w_0) \theta_0 x^{n-\sigma}) \leq n - \sigma - 1$, then $\deg \Lambda_{n+1} \leq n + 1$, $n \geq \sigma$.

(B) $\langle v \rangle_0 \neq 0$. By Lemma 3.1, $\deg \Phi = t \leq \sigma + 2$, $\deg \Psi = p \leq \sigma + 1$ and $\deg \hat{B} = \tilde{r} \leq \sigma + 2$. Besides, $\deg((\Phi w_0) \theta_0 x^{n-\sigma}) = n - \sigma - 1$, because we have $(\Phi w_0) = \lambda^{-1}(v)_0 \neq 0$. Therefore, since $\deg D_u(x^{n-\sigma}) = n - \sigma - 1$ and $\deg(\hat{B} w_0) \theta_0 x^{n-\sigma} \leq n - \sigma - 1$, we obtain $\deg \Lambda_{n+1} \leq n + 1$, $n \geq \sigma$. Integral Transforms and Special Functions
By inserting (3.18) in (3.17), we obtain

\[(n + (u)_{0} + 1)\langle v, x^{n-\sigma} B_{n}^{[1]}(x; u) \rangle = -\lambda \langle w_{0}, \Lambda_{n+1} B_{n+1}(x) \rangle, \quad n \geq \sigma. \quad (3.20)\]

Denoting by \(\rho_{n}\) the coefficient of the monomial \(x^{n+1}\) in \(\Lambda_{n+1}(x)\), \(n \geq \sigma\), and using (3.19) we obtain

\[
\rho_{n} = (n - \sigma) \frac{\Phi^{(\sigma+2)}(0)}{(\sigma + 2)!} - \frac{\Psi^{(\sigma+1)}(0)}{(\sigma + 1)!} + (v)_{0} \frac{\hat{B}^{(\sigma+2)}(0)}{(\sigma + 2)!\lambda}, \quad n \geq \sigma. \quad (3.21)
\]

Hence, (3.20) can be written as

\[
(n + (u)_{0} + 1)\langle v, x^{n-\sigma} B_{n}^{[1]}(x; u) \rangle = -\lambda \rho_{n} \langle w_{0}, B_{n+1}^{2} \rangle, \quad n \geq \sigma. \quad (3.22)
\]

For \(n = r\) in (3.22), where \(r\) is given by (3.2), \((r \geq \sigma)\), since we have \(\langle v, x^{r-\sigma} B_{r}^{[1]}(x; u) \rangle \neq 0\), then \(\rho_{r} \neq 0\) and so that deg \(\Lambda_{r+1} = r + 1\). Thus,

\[
\sigma = \max(t - 2, p - 1). \quad (3.23)
\]

Indeed, first note that we have \(\sigma \geq \max(t - 2, p - 1)\). If we assume that \(\sigma > \max(t - 2, p - 1)\), then the analysis of the degrees of the polynomial \(\Lambda_{r+1}\), yields deg \(\Lambda_{r+1} \leq r\). This is a contradiction. Hence, (3.23) holds.

As a consequence, we can analyse two situations.

When \(t = \sigma + 2\), then

\[
\rho_{n} = n - \sigma - \frac{\Psi^{(\sigma+1)}(0)}{(\sigma + 1)!} + (v)_{0} \frac{\hat{B}^{(\sigma+2)}(0)}{(\sigma + 2)!\lambda}, \quad n \geq \sigma. \quad (3.24)
\]

The MPS \(\{B_{n}^{[1]}(x; u)\}_{n \geq 0}\) is strictly quasi-orthogonal of order \(\sigma\) if and only if

\[-\Psi^{(\sigma+1)}(0)/(\sigma + 1)! + (v)_{0}(\hat{B}^{(\sigma+2)}(0))/(\sigma + 2)!\lambda) \neq -n, \quad n \geq 0.\]

When \(t < \sigma + 2\), then

\[
\rho_{n} = -\frac{\Psi^{(\sigma+1)}(0)}{(\sigma + 1)!} + (v)_{0} \frac{\hat{B}^{(\sigma+2)}(0)}{(\sigma + 2)!\lambda} = \rho_{r} \neq 0, \quad n \geq \sigma. \quad (3.25)
\]

Thus, the MPS \(\{B_{n}^{[1]}(x; u)\}_{n \geq 0}\) is strictly quasi-orthogonal of order \(\sigma\).

(ii) \(\Rightarrow\) (i). We have \((n + (u)_{0} + 1)\langle \Phi w_{0}, x^{m} B_{n}^{[1]}(x; u) \rangle = -\langle D_{0}(x^{m}\Phi w_{0}), B_{n+1} \rangle\), for all non-negative integers \(m\) and \(n\). From (2.2) and (3.15) as above, we have \(D_{0}(x^{m}\Phi w_{0}) = \Lambda_{m+\sigma+1} w_{0}\), with \(\Lambda_{m+\sigma+1}(x) = -x^{m}\Psi(x) + \Phi(x) D_{0,\tau}(x^{m}) + (\Phi w_{0}) \theta_{0} x^{m} \hat{B}(x) - ((\hat{B} w_{0}) \theta_{0} x^{m}) \Phi(x)\), and where deg \(\Lambda_{m+\sigma+1} \leq m + \sigma + 1, \quad m \geq 0\). Thus,

\[
(n + (u)_{0} + 1)\langle \Phi w_{0}, x^{m} B_{n}^{[1]}(x; u) \rangle = -\langle w_{0}, \Lambda_{m+\sigma+1} B_{n+1} \rangle. \quad (3.26)
\]

Using the orthogonality of \(\{B_{n}\}_{n \geq 0}\), we obtain

\[
\langle \Phi w_{0}, x^{m} B_{n}^{[1]}(x; u) \rangle = 0, \quad 0 \leq m \leq n - \sigma - 1, \quad n \geq \sigma + 1. \quad (3.27)
\]

For \(m = n - \sigma\), in (3.26) we get

\[
(n + (u)_{0} + 1)\langle \Phi w_{0}, x^{n-\sigma} B_{n}^{[1]}(x; u) \rangle = -\rho_{n} \langle w_{0}, B_{n+1}^{2} \rangle, \quad n \geq \sigma. \quad (3.28)
\]

where \(\rho_{n} = (n - \sigma)(\Phi^{(\sigma+2)}(0))/(\sigma + 2)! - (\Psi^{(\sigma+1)}(0))/(\sigma + 1)! + (\Phi w_{0})(\hat{B}^{(\sigma+2)}(0))/(\sigma + 2)!\), \(n \geq \sigma\).
Note that there exists an integer \( r \geq \sigma \) such that \( \langle \Phi w_0, x^{r-\sigma} B_n^{[1]}(x; u) \rangle \neq 0 \). Otherwise, \( \langle \Phi w_0, x^{n-\sigma} B_n^{[1]}(x; u) \rangle = 0, \ n \geq \sigma. \) Then, according to (3.28) we get \( \rho_n = 0, \ n \geq \sigma. \) This requires that \( t \leq \sigma + 1 = p \) and \(- (\sigma + 2) \Psi^{(\sigma+1)}(0) + (\Phi w_0) B^{(\sigma+2)}(0) = 0.\) So, \( \langle \Phi w_0, x^{m} B_n^{[1]}(x; u) \rangle = 0, \ 0 \leq m \leq n - \sigma, \ n \geq \sigma. \) In this way, there exists \((l, r) \in \mathbb{N}^2, \) with \( 1 \leq l \leq \sigma \) and \( r \geq l - 1, \) such that

\[
\langle \Phi w_0, x^{l-\sigma} B_n^{[1]}(x; u) \rangle = 0, \ 0 \leq m \leq n - \sigma + l - 1, \ n \geq \sigma - l + 1, \tag{3.29}
\]

\[
\langle \Phi w_0, x^{r-\sigma} B_n^{[1]}(x; u) \rangle \neq 0. \tag{3.30}
\]

Otherwise, \( \langle \Phi w_0, x^{m} B_n^{[1]}(x; u) \rangle = 0, \ 0 \leq m \leq n. \) In particular, for \( m = 0, \ \langle \Phi w_0, B_n^{[1]}(x; u) \rangle = 0, \ n \geq 0, \) i.e. \( \Phi w_0 = 0, \) where \( \Phi \neq 0. \) This contradicts the quasi-definiteness of the linear functional \( w_0. \)

Hence, from (3.29) and (3.30), \( \{ B_n^{[1]}(x; u)\}_{n \geq 0} \) will be quasi-orthogonal of order \( \sigma' = \sigma - l = p - 1 - l \) with respect to \( v = \Phi w_0. \)

In addition, \( (p + (u)) \langle \Phi w_0, B_{p-1}^{[1]}(; u) \rangle = \langle \Phi w_0, \Phi(\Phi w_0), B_{p-1}^{[1]}(x; u) \rangle = 0 \) for \( \Phi w_0, \) \( B_n^{[1]}(x; u) = 0, \ n \geq \sigma - l + 1, \) due to (3.29), where \( m = 0. \) Since \( \langle \Phi w_0, B_n^{[1]}(x; u) \rangle \neq 0, \) then \( \sigma - l + 1 \geq p. \) But \( \sigma = p - 1 \) and, as a consequence, \( p - l \geq p, \) i.e. \( l \leq 0. \) This contradicts the fact that \( l \geq 1. \)

Finally, there exists an integer \( r \geq \sigma \) such that \( \langle \Phi w_0, x^{m} B_n^{[1]}(x; u) \rangle = 0, \ 0 \leq m \leq n - \sigma - 1 \) and \( \langle \Phi w_0, x^{r-\sigma} B_n^{[1]}(x; u) \rangle \neq 0. \) Hence, (i) holds.

\[\square\]

4. Another characterization of a \( D_n \)-semiclassical MOPS

The following result allows us to characterize the \( D_n \)-semiclassical monic orthogonal polynomial sequence by a first structure relation.

**Theorem 4.1** Let \( u \in \mathbb{P}' \) and \( \{ B_n\}_{n \geq 0} \) be an MOPS with respect to \( w_0. \) The following statements are equivalent.

(i) \( \{ B_n\}_{n \geq 0} \) is \( D_n \)-semiclassical.

(ii) There exists \((r, \sigma) \in \mathbb{N}^2, \) with \( 0 \leq \sigma \leq r, \) such that

\[\Phi(x) B_n^{[1]}(x; u) = \sum_{v=\sigma}^{n+r} \lambda_{n,v} B_v(x), \ n \geq \sigma, \tag{4.1}\]

\[\lambda_{r,r-\sigma} \neq 0. \tag{4.2}\]

**Proof** (i) \( \Rightarrow \) (ii). First, we always have \( \Phi B_n^{[1]}(; u) = \sum_{v=0}^{n+r} \lambda_{n,v} B_v, \ n \geq 0, \) with \( t = \deg \Phi \) and \( \lambda_{n,v} = \langle w_0, B_n^{[1]}(.; u) \rangle \langle w_0, B_v^{[1]}(.; u) \rangle^{-1}, \ 0 \leq v \leq n + t, \ n \geq 0. \) But, \( \{ B_n^{[1]}(x; u)\}_{n \geq 0} \) is quasi-orthogonal of order \( \sigma = \max(t - 2, p - 1) \) with respect to \( \Phi w_0, \) according to Definition 2.7 and Theorem 3.2. Therefore, there exists an integer \( r \geq \sigma \) such that \( \langle \Phi w_0, B_r B_n^{[1]}(x; u) \rangle = 0, \ 0 \leq v \leq n - \sigma - 1, \ n \geq \sigma + 1 \) and \( \langle \Phi w_0, B_{r-\sigma} B_n^{[1]}(x; u) \rangle \neq 0. \) Thus, we have \( \lambda_{n,v} = 0, \ 0 \leq v \leq n - \sigma - 1, \ n \geq \sigma + 1, \) and \( \lambda_{r,r-\sigma} \neq 0. \) Hence, (4.1) and (4.2) hold, with \( \lambda_{r,r-\sigma} = \langle w_0, B_{r-\sigma}^{[1]}(.; u) \rangle^{-1} \langle \Phi w_0, B_{r-\sigma} B_{r}^{[1]}(x; u) \rangle \neq 0. \)

(ii) \( \Rightarrow \) (i). From the assumption and the orthogonality of \( \{ B_n\}_{n \geq 0}, \) we get \( \langle \Phi w_0, B_m B_n^{[1]}(.; u) \rangle = \sum_{v=\sigma}^{n+r} \lambda_{n,v} \langle w_0, B_m B_v (.; u) \rangle \delta_{m,v}, \ n \geq \sigma. \) Then, \( \langle \Phi w_0, B_m B_n^{[1]}(.; u) \rangle = 0, \ 0 \leq m \leq n - \sigma - 1, \ n \geq \sigma + 1 \) and \( \langle \Phi w_0, B_{r-\sigma} B_n^{[1]}(.; u) \rangle = \lambda_{r,r-\sigma} \langle w_0, B_{r-\sigma}^{[1]} \rangle \neq 0. \) Thus, \( \{ B_n^{[1]}(x; u)\}_{n \geq 0} \) is quasi-orthogonal of order \( \sigma \) with respect to \( \Phi w_0. \) \[\square\]
The following result shows that any $D_n$-semiclassical MOPS is a $D$-Laguerre–Hahn MOPS.

**Theorem 4.2** Let $u \in \mathbb{P}_M'$ and $\{B_n\}_{n \geq 0}$ be an MOPS with respect to $w_0$. The following statements are equivalent.

(i) $\{B_n\}_{n \geq 0}$ is a $D_n$-semiclassical MOPS.

(ii) There exists $(\Phi, \Psi, \hat{B}) \in \mathbb{P}^3$, $\Phi$ monic, $\deg \Phi = t$, $\deg \Psi = p \geq 1$, $\deg \hat{B} \leq k + \sigma + 2$, with $k = \min\{m \geq 0 : (\Phi w_0)_m \neq 0\}$, and $\sigma = \max(t - 2, p - 1)$, such that $\hat{B} w_0 \in \mathbb{P}_M'$, and where $w_0$ satisfies

$$
(\Phi w_0)' + \Psi w_0 - (\Phi \hat{B})(x^{-1} w_0^2) = 0,
$$

where

$$
\Psi_1(x) = \Psi(x) + \Phi(x)(w_0 \theta_0 \hat{B})(x) + \hat{B}(x)(w_0 \theta_0 \Phi)(x).
$$

Proof (i) $\Rightarrow$ (ii). Assuming (i), By Theorem 3.2(ii), there exists $(\Phi, \Psi, \hat{B}) \in \mathbb{P}$, $\Phi$ monic, $\deg \Phi = t$, $\deg \Psi = p \geq 1$, $\deg \hat{B} \leq k + \sigma + 2$, with $\sigma = \max(t - 2, p - 1)$ and $k = \min\{m \geq 0 \mid (\Phi w_0)_m \neq 0\}$, such that $D_n(\Phi w_0) + \Psi w_0 = 0$, with $u = \hat{B} w_0$. But, $D_n(\Phi w_0) = (\Phi w_0)' - x^{-1} u(\Phi w_0)$, then

$$
(\Phi w_0)' - x^{-1}((\hat{B} w_0)(\Phi w_0)) + \Psi w_0 = 0.
$$

Using (1.4) we can write

$$
(\hat{B} w_0)(\Phi w_0) = \Phi \hat{B} w_0^2 - x F w_0,
$$

where

$$
F(x) := \Phi(x)(w_0 \theta_0 \hat{B})(x) + \hat{B}(x)(w_0 \theta_0 \Phi)(x).
$$

From (4.6), (1.3) and (1.6), we get

$$
x^{-1}((\hat{B} w_0)(\Phi w_0)) = (\Phi \hat{B})x^{-1} w_0^2 - F w.
$$

Then, (4.5) becomes $(\Phi w_0)' + (F + \Psi) w_0 - (\Phi \hat{B})(x^{-1} w_0^2) = 0$. Hence, (4.3) holds, with $\Psi_1(x) = F(x) + \Psi(x)$.

(ii) $\Rightarrow$ (i). Assuming (ii), let $F(x) = \Phi(x)(w_0 \theta_0 \hat{B})(x) + \hat{B}(x)(w_0 \theta_0 \Phi)(x)$. Setting $u = \hat{B} w_0$, $\Psi(x) = \Psi_1(x) - F(x)$ and using (4.3), (4.4) and (4.8), we get $D_n(\Phi w_0) + \Psi w_0 = (\Phi \hat{B})(x^{-1} w_0^2) - x^{-1}((\hat{B} w_0)(\Phi w_0)) - F w = 0$. Hence, by Theorem 3.2(i) holds.

**Corollary 4.3** A $D$-Laguerre–Hahn linear functional $w_0$ satisfying $(\Phi w_0)' + (\Psi_1 w_0) + B(x^{-1} w_0^2) = 0$, with $(\Phi, \Psi_1, B) \in \mathbb{P}^3$ and $\Phi$ monic, is $D_n$-semiclassical if and only if the following conditions hold.

(i) $\Phi(x)$ divides $B(x)$. We will write $B(x) = -\Phi(x) \hat{B}(x)$.

(ii) $\deg \hat{B} \leq k + \sigma + 2$, where $k = \min\{m \geq 0 \mid (\Phi w_0)_m \neq 0\}$, $\sigma = \max(\deg(\Phi) - 2, \deg(\Psi) - 1)$, and $\Psi = \Psi_1 - \Phi(w_0 \theta_0 \hat{B}) - \hat{B}(w_0 \theta_0 \Phi)$.

(iii) $\hat{B} w_0 = 0 \neq -n, n \geq 1$.

In this case, $u = \hat{B} w_0$.

As an example, let us consider the singular $D$-Laguerre–Hahn MOPS of class zero analogous to the Laguerre’s one which we denote by $\{l_n(\cdot; \xi)\}_{n \geq 0}$ with $\xi = (\alpha, \lambda, \rho)$ (see [1,4]). We will denote by $\ell(\xi)$ the corresponding normalized linear functional. In Table 1, we summarize the characteristic elements of the MOPS $\{l_n(\cdot; \xi)\}_{n \geq 0}$.
Table 1. The singular Laguerre–Hahn sequences of class zero analogous to Laguerre’s ones with parameter $\xi = (\alpha, \lambda, \rho)$.

<table>
<thead>
<tr>
<th>$\Phi(x)$</th>
<th>$\Psi(x)$</th>
<th>$\beta_0$</th>
<th>$\beta_{n+1}$</th>
<th>$\gamma_n$</th>
<th>$C_0(x)$</th>
<th>$D_0(x)$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, \Psi_1(x) = \frac{\xi}{x+\alpha-1}$</td>
<td>$x^2 + (2(1-\alpha) - \lambda)x + \alpha(\alpha + \lambda - 1) - \rho$</td>
<td>$\alpha + \lambda - 1$</td>
<td>$2n + \alpha + 1$, $n \geq 0$</td>
<td>$\gamma_n = \rho$, $\gamma_{n+1} = n(n + \alpha)$, $n \geq 1$</td>
<td>$x - \alpha$, $C_{n+1}(x) = -x + 2n + \alpha$, $n \geq 0$</td>
<td>$D_0(x) = 0$, $D_{n+1}(x) = -1$, $n \geq 0$</td>
<td>$\lambda \in \mathbb{C}$, $\rho \neq 0$, $\alpha \neq -n$, $n \geq 1$</td>
</tr>
</tbody>
</table>

Using Corollary 4.3, the condition (i) is equivalent to $\rho = \alpha(\alpha + \lambda - 1)$. Then, $\hat{B}(x) = -x + 2(\alpha - 1) + \lambda$, $\Phi(x) = x - (\alpha - 1 + \lambda)$, and $\sigma = 0$. Besides, $(xw_0)_0 = \alpha + \lambda - 1 \neq 0$. $\rho = \alpha(\alpha + \lambda - 1) \neq 0$, then $k = 0$. Therefore, (ii) holds because $\deg \hat{B} = 1 \leq k + \sigma + 2 = 2$. Finally, since $(\hat{B}\ell(\xi))_0 = \alpha - 1$, the condition (iii) is equivalent to $\alpha \neq -n$, $n \geq 0$. Thus, the MOPS $\{l_n(\cdot; \xi^*)\}_{n \geq 0}$, where $\xi^* = (\alpha, \lambda, \alpha(\alpha + \lambda - 1))$, with $\alpha \neq -n$, $n \geq 0$, and $\alpha + \lambda - 1 \neq 0$ is $D_{\ell(\xi^*)}$-semiclassical, with $\mathcal{U}(\xi^*) = \hat{B}\ell(\xi^*)$ and $\hat{B}(x) = -x + 2(\alpha - 1) + \lambda$. The linear functional $\ell(\xi^*)$ satisfies: $D_{\ell(\xi^*)}(x\ell(\xi^*)) + (x - \alpha - \lambda + 1)\ell(\xi^*) = 0$. Also, $\ell(\xi^*)$ is a D-Laguerre–Hahn linear functional of class zero, since it satisfies: $(x\ell(\xi^*))' + (x - \alpha - 1)\ell(\xi^*) - x - 2(\alpha - 1) + \lambda(x^{-1}(\ell(\xi^*))^2) = 0$.

In summary, we have the following.

Since the D-Laguerre–Hahn MOPS $\{l_n(\cdot; \xi^*)\}_{n \geq 0}$ satisfies an FSR, then from Table 2 and (1.9), we get

$$x l_{n+1}'(x; \xi^*) + x \hat{B}(x) l_n^{(1)}(x; \xi^*) = (-x + n + \alpha) l_{n+1}(x; \xi^*) + n(n + \alpha) l_n(x; \xi^*), \quad n \geq 1. \quad (4.9)$$

But, from the orthogonality of $\{l_n(\cdot; \xi^*)\}_{n \geq 0}$ with respect to $\ell(\xi^*)$, we get

$$(\mathcal{U}(\xi^*) \theta_0 l_{n+1}(\cdot; \xi^*))(x) = l_{n+1}(x; \xi^*) + \hat{B}(x)(\ell(\xi^*) \theta_0 l_{n+1}(\cdot; \xi^*))(x),$$

$$= l_{n+1}(x; \xi^*) + \hat{B}(x) l_n^{(1)}(x; \xi^*), \quad n \geq 0.$$  

Then, (4.9) becomes $x l_n^{(1)}(x; \xi^*) = l_{n+1}(x; \xi^*) + n l_n(x; \xi^*)$, $n \geq 1$, where $\{l_n^{(1)}(\cdot; \xi^*)\}_{n \geq 0}$ is the $D_{\ell(\xi^*)}$-derivative sequence of $\{l_n(\cdot; \xi^*)\}_{n \geq 0}$, defined by

$$(n + \alpha) l_n^{(1)}(x; \xi^*) = l_{n+1}'(x; \xi^*) + (\mathcal{U}(\xi^*) \theta_0 l_{n+1}(\cdot; \xi^*))(x), \quad n \geq 0.$$  

In addition, it is easy to see that $x l_0^{(1)}(x; \xi^*) = l_1(x; \xi^*) + (\alpha + \lambda - 1) l_0(x; \xi^*)$. Hence, the FSR becomes

$$x l_0^{(1)}(x; \xi^*) = l_1(x; \xi^*) + \vartheta_0 l_0(x; \xi^*), \quad n \geq 0, \quad (4.10)$$

where $\vartheta_n = n$, $n \geq 1$ and $\vartheta_0 = \alpha + \lambda - 1$. 

By (4.10) and the orthogonality of $\{l_n(\cdot; \xi^*)\}_{n \geq 0}$ with respect to $\ell(\xi^*)$, we get

$$
\langle x \ell(\xi^*), x^m l_n^{[1]}(x; \xi^*) \rangle = 0, \quad 0 \leq m \leq n - 1, \quad n \geq 1,
$$

$$
\langle x \ell(\xi^*), x^n l_n^{[1]}(x; \xi^*) \rangle = \partial_n \ell(\xi^*), l_n^2(x; \xi^*) \rangle \neq 0, \quad n \geq 0.
$$

Equivalently, \((\alpha + \lambda - 1)^{-1}x \ell(\xi^*), l_n^{[1]}(\cdot; \xi^*)l_n^{[1]}(\cdot; \xi^*) = r_n^{[1]}(\xi^*) \delta_{n,m}, \ n, m \geq 0, \) with $r_n^{[1]}(\xi^*) = (\alpha + \lambda - 1)^{-1} \partial_n \ell(\xi^*), l_n^2(x; \xi^*) = n! \Gamma(\alpha)^{-1} \Gamma(n + \alpha) \neq 0, \ n \geq 0$.

Thus, $\{l_n^{[1]}(\cdot; \xi^*)\}_{n \geq 0}$ is orthogonal with respect to $\ell_0^{[1]}(\xi^*) = (\alpha + \lambda - 1)^{-1}x \ell(\xi^*)$. The MOPS $\{l_n^{[1]}(\cdot; \xi^*)\}_{n \geq 0}$ satisfies the following TTRR:

$$
l_0^{[1]}(x; \xi^*) = 1, \quad l_n^{[1]}(x; \xi^*) = x - \beta_n^{[1]},
$$

$$
l_{n+1}^{[1]}(x; \xi^*) = (x - \tilde{\beta}_{n+1}) l_n^{[1]}(x; \xi^*) - \gamma_n l_n^{[1]}(x; \xi^*), \quad n \geq 0, \quad (4.11)
$$

where $\gamma_n^{[1]} = r_n^{[1]}(\xi^*) r_n^{[1]}(\xi^*)^{-1} = (n + 1)(n + \alpha) \neq 0, \ n \geq 0$, and from (4.10) and the orthogonality of $\{l_n(\cdot; \xi^*)\}_{n \geq 0}$ with respect to $\ell(\xi^*)$, we get

$$
\beta_n^{[1]} = \frac{\langle \ell(\xi^*), [l_{n+1}(\cdot; \xi^*)]^2 \rangle + \partial_n \langle \ell(\xi^*), [l_n(\cdot; \xi^*)]^2 \rangle}{\partial_n \langle \ell(\xi^*), (l_n(\cdot; \xi^*))^2 \rangle} = \gamma_n l_n^{[1]} + \partial_n, \quad n \geq 0.
$$

Thus, $\beta_n^{[1]} = 2\alpha + \lambda - 1, \ \beta_{n+1}^{[1]} = 2n + \alpha + 2, \ n \geq 0$.

Note that $\{l_n^{[1]}(\cdot; \xi^*)\}_{n \geq 0}$ is a D-Laguerre–Hahn MOPS of class zero. More precisely, it is a non-singular D-Laguerre–Hahn MOPS analogous of the Laguerre’s one [1,4]. Recall that the non-singular D-Laguerre–Hahn MOPS analogous to the Laguerre’s polynomial sequence denoted by $\{L_n(\cdot; \varpi)\}_{n \geq 0}$ with $\varpi = (\lambda, \rho, \tau, \alpha)$, satisfies the following TTRR:

$$
L_0(x; \varpi) = 1, \quad L_1(x; \varpi) = x - \tilde{\beta}_0,
$$

$$
L_{n+2}(x; \varpi) = (x - \tilde{\beta}_{n+1}) L_{n+1}(x; \varpi) - \gamma_{n+1} L_n(x; \varpi), \quad n \geq 0,
$$

where

$$
\tilde{\beta}_0 = 2\tau + \alpha + \lambda + 1, \quad \tilde{\beta}_{n+1} = 2(n + \tau + 1) + \alpha + 1, \ n \geq 0,
$$

$$
\gamma_1 = \rho(\tau + 1)(\tau + \alpha + 1), \quad \gamma_{n+1} = (n + \tau + 1)(n + \tau + \alpha + 1), \ n \geq 1,
$$

with $\tau + \alpha \neq -(n + 1)$ and $\tau \neq -(n + 1), \ n \geq 0$.

As a straightforward consequence, we get $l_n^{[1]}(x; \xi^*) = L_n(x; \varpi^*), \ n \geq 0$, where $\varpi^* = (\alpha + \lambda - 1, \ 0, \ \alpha - 1)$.

We conclude our work stating two finite-type relations linking $\{l_n(\cdot; \xi^*)\}_{n \geq 0}$ to $\{L_n(\cdot; \varpi^*)\}_{n \geq 0}$, which are known in the literature as second structure relations (see [10,12]). By inserting (4.10) in (3.3), where $B_n(x) = l_n(x; \xi^*), \ n \geq 0$, and using the fact that $\langle U(\xi^*), l_{n+1}(x; \xi^*) \rangle = -\delta_{n,0}, \ n \geq 0$, we obtain

$$
l_{n+1}(x; \xi^*) + \nu_n l_n(x; \xi^*) = L_{n+1}(x; \varpi^*) + \frac{(n + 1)(2n + \alpha + 1)}{n + \alpha + 1} L_n(x; \varpi^*)
$$

$$
+ \frac{(n + \alpha - 1)}{n + \alpha + 1} L_{n-1}(x; \varpi^*) + (\alpha + 1)^{-1} \delta_{n,0}, \ n \geq 0,
$$

where $\nu_n = n(n + \alpha)(n + \alpha + 1)^{-1}$.

Finally, substituting (4.11) into (3.3) where $B_n(x) = l_n(x; \xi^*), \ n \geq 0$, and using the fact that $\langle U(\xi^*), l_{n+1}(x; \xi^*) \rangle = -\delta_{n,0}, \ n \geq 0$, we obtain

$$
l_n(x; \xi^*) = L_n(x; \varpi^*) + (n + \alpha - 1) L_{n-1}(x; \varpi^*), \ n \geq 0, \ L_{-1}(x; \varpi^*) = 0.
$$
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