INVERSE FINITE-TYPE RELATIONS BETWEEN SEQUENCES OF POLYNOMIALS

by

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Abstract

Let $\phi$ be a monic polynomial, with $\deg \phi = t \geq 0$. We say that there is a finite-type relation between two monic polynomial sequences $\{B_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ with respect to $\phi$, if there exists $(s, r) \in \mathbb{N}^2$, $r \geq s$, such that

$$\phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_{\nu}(x), \quad n \geq s, \text{ with } \lambda_{r,r-s} \neq 0.$$  

The corresponding inverse finite-type relation of $(\ast)$ consists in a finite-type relation as follows:

$$\Omega_n^*(x; n) B_n(x) = \sum_{\nu=n-t}^{n+s} \theta_{n,\nu}^* Q_{\nu}(x), \quad n \geq t, \text{ with } \theta_{r,t+r}^* \neq 0,$$

where $\deg \Omega_n^*(x; n) = s$, $n \geq t$. When the orthogonality of the two previous sequences is assumed, the inverse finite-type relation is always possible [11]. This work essentially studies the case when only the sequence $\{B_n\}_{n \geq 0}$ is orthogonal. In fact, we find necessary and sufficient conditions leading to inverse finite-type relations. In particular, the structure relation characterizing semi-classical sequences is a special case of the general situation. Some examples will be analyzed.

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Resumen

Sea $\phi$ un polinomio mónico, con $\deg \phi = t \geq 0$. Decimos que hay relación de tipo finito entre dos sucesiones de polinomios mónicos $\{B_n\}_{n \geq 0}$ y $\{Q_n\}_{n \geq 0}$ con respecto a $\phi$, si existe $(s, r) \in \mathbb{N}^2$, $r \geq s$, tal que

$$
\phi(x)Q_n(x) = \sum_{s=n-r}^{n-t} \lambda_{n,r}B_r(x), \quad n \geq s, \quad \text{con} \quad \lambda_{r,r-s} \neq 0. \quad (*)
$$

La correspondiente relación de tipo finito de $(*)$ consiste en una relación de tipo finito como sigue:

$$
\Omega^*_n(x; n)B_n(x) = \sum_{t=n-r}^{n+s} \theta^*_tQ_n(x), \quad n \geq t, \quad \text{con} \quad \theta^*_{r+t,r} \neq 0,
$$

donde $\deg \Omega_n^*(x; n) = s, \ n \geq t$. Cuando se supone la ortogonalidad de las dos sucesiones previas, la relación de tipo finito inversa siempre es posible [11]. En este trabajo se estudia el caso en que solo la sucesión $\{B_n\}_{n \geq 0}$ es ortogonal. De hecho, encontramos condiciones necesarias y suficientes que conducen a relaciones de tipo finito inversas. En particular, la relación de estructura que caracteriza a las sucesiones semiclásicas es un caso especial de la situación general. Se estudian varios ejemplos.

Palabras clave: Relaciones de tipo finito, relaciones de recurrencia, polinomios ortogonales, polinomios semi clásicos.

1. Introduction and background

Let $\mathbb{P}$ be the linear space of complex polynomials in one variable and $\mathbb{P}'$ its topological dual space. We denote by $\langle u, f \rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of $u$ with respect to the polynomial sequence $\{x^n\}_{n \geq 0}$.

We will introduce some useful operations in $\mathbb{P}'$. For any linear functional $u$ and any polynomial $h$, let $Du = u'$ and $hu$ be the linear functionals defined by duality

$$
\langle u', f \rangle := -\langle u, f' \rangle, \quad f \in \mathbb{P},
$$

$$
\langle hu, f \rangle := \langle u, hf \rangle, \quad f, h \in \mathbb{P}.
$$

Let $\{B_n\}_{n \geq 0}$ be a monic polynomial sequence (MPS), $\deg B_n = n, \ n \geq 0$, and $\{u_n\}_{n \geq 0}$ its dual sequence, $u_n \in \mathbb{P}'$, $n \geq 0$, defined by $\langle u_n, B_m \rangle := \delta_{n,m}, \ n, \ m \geq 0$, where $\delta_{n,m}$ is the Kronecker symbol.

Let recall the following results [11].

Lemma 1.1. For any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent.

i) $\langle u, B_{m-1} \rangle \neq 0$, $\langle u, B_m \rangle = 0$, $n \geq m$.

ii) There exist $\lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq m - 1$, $\lambda_{m-1} \neq 0$, such that $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n \geq 0}$ of the sequence $\{B_n^{[1]}\}_{n \geq 0}$, where $B_n^{[1]}(x) = (n + 1)^{-1}B_{n+1}(x)$, $n \geq 0$, satisfies

$$
\langle u_n^{[1]} \rangle = -(n + 1)u_{n+1}, \ n \geq 0. \quad (1.1)
$$

Definition 1.2. The linear functional $u$ is said to be regular if there exists a monic polynomial sequence $\{B_n\}_{n \geq 0}$ such that

$$
\langle u, B_n B_m \rangle = b_n \delta_{n,m}, \ n, \ m \geq 0, \quad (1.2)
$$

where

$$
b_n = \langle u, B_n^2 \rangle \neq 0, \ n \geq 0. \quad (1.3)
$$

Then the sequence $\{B_n\}_{n \geq 0}$ is said to be orthogonal (MOPS) with respect to $u$.

As a straightforward consequence we get

- The linear functional can be represented by $u = (u)_0 u_0$, and the following relations hold

$$
u_n = b_n^{-1} B_n u, \ n \geq 0. \quad (1.4)
$$
• The sequence \( \{B_n\}_{n \geq 0} \) satisfies the three-term recurrence relation

\[
B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0,
\]

\[
B_1(x) = x - \beta_0, \quad B_0(x) = 1,
\]

(1.5)

where \( \gamma_{n+1} \neq 0, n \geq 0 \) (see [4]).

In the sequel and under the assumption of the previous definition, we need to put

\[
b_{n,m}^l = b_{n}^{-1}(x, x^\nu B_t B_n), \quad (n, \nu, m) \in \mathbb{N}^3.
\]

(1.6)

In particular, one has

\[
b_{n,m}^l = \begin{cases}
0, & \text{if } \nu + m < n, \quad 0 \leq m < n, \quad \nu \geq 0, \\
(b_{n,0, 1}/b_{n}), & \text{if } \nu = n - m, \quad 0 \leq m \leq n.
\end{cases}
\]

Let \( \phi \) be a monic polynomial, with \( \deg \phi \geq t \geq 0 \). For any MPS \( \{B_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) with dual sequences \( \{u_n\}_{n \geq 0} \) and \( \{v_n\}_{n \geq 0} \) respectively, the following formula always holds

\[
\phi(x)Q_n(x) = \sum_{\nu = n-t}^{n+t} \lambda_{n,\nu} B_{\nu}(x), \quad n \geq 0,
\]

(1.7)

where \( \lambda_{n,\nu} = \langle u_{\nu}, \phi Q_n \rangle, \quad 0 \leq \nu \leq n + t, \quad n \geq 0 \).

**Definition 1.3.** ([12]) If there exists an integer \( s \geq 0 \) such that

\[
\phi(x)Q_n(x) = \sum_{\nu = n-s}^{n+t} \lambda_{n,\nu} B_{\nu}(x), \quad n \geq s,
\]

(1.8)

and

\[
\exists \ t \geq s, \lambda_{r, r - s} \neq 0,
\]

(1.9)

then, we shall say that (1.8) – (1.9) gives a finite-type relation between \( \{B_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \), with respect to \( \phi \).

When instead of (1.9), we take

\[
\lambda_{n, n-s} \neq 0, \quad n \geq s,
\]

(1.9')

we shall say that (1.8) – (1.9') is a strictly finite-type relation.

The corresponding inverse finite-type relation of (1.8) – (1.9) consists in establishing, whenever it is possible, a finite-type relation between \( \{Q_n\}_{n \geq 0} \) and \( \{B_n\}_{n \geq 0} \), as follows

\[
\Omega_{n}(x; n)B_n(x) = \sum_{\nu = n-t}^{n+s} \theta_{n,\nu} Q_{\nu}(x), \quad n \geq t,
\]

(1.10)

\[
\theta_{r, t, r}^{\nu} \neq 0, \quad \text{where } \{\Omega_{n}(x; n)\}_{n \geq t} \text{ is a MPS, } \deg \Omega_{n}(x; n) = s, \quad n \geq t, \quad \text{and}
\]

\[
(\theta^{\nu}_{t, n, t} + s)_{\nu = n-t}, \quad n \geq t,
\]

(1.11)

a system of complex numbers (SCN), with \( \theta_{n, n+s}^{\nu} = 1, \quad n \geq t \).

When both two sequences are orthogonal, the inverse relation is always possible. In this case, the polynomials \( \Omega_{n}(x; n), \quad n \geq 0, \) are independent of \( n \), (see [12], Proposition 2.4). As a current example, we can mention the two structure relations characterizing the classical polynomials, (Hermite, Laguerre, Bessel, Jacobi, see [11]), which could solely be two inverse finite-type relations.

In other studies, we find several situations where one of the two sequences is orthogonal. For example, the structure relations characterizing semi-classical sequences associated with Hahn’s operators \( L_{q, \omega} \), with parameters \( q \) and \( \omega \), [9]. The Coherent pairs and Diagonal sequences are also examples of finite type-relations [7, 12, 13, 14]. But the inverse relations corresponding to other finite-type relations are not yet considered.

The paper essentially gives a necessary and sufficient condition allowing the existence of the inverse finite-type relations when the orthogonality of the sequence \( \{B_n\}_{n \geq 0} \) is assumed. From now on, it would be necessary to study the case where the sequence \( \{Q_n\}_{n \geq 0} \) is orthogonal. It would be very useful to deal with many other situations like General Coherent pairs, see [6, 8] in the framework of Sobolev inner products.

## 2. A basic result

We use this section to introduce some auxiliary result for the proof of the main theorem in section 3.

**Lemma 2.1.** Suppose \( \{B_n\}_{n \geq 0} \) is a MOPS and \( \{Q_n\}_{n \geq 0} \) fulfills (1.8) – (1.9), where \( t = 0 \) and \( s \geq 1 \).

For any SCN \( \{\theta_{n,\nu}\}_{\nu = n-t}, \quad n \geq 0, \) \( \theta_{n, n+s} = 1, \quad n \geq 0, \) and \( \theta_{r, r} \neq 0, \) there exist a unique MPS \( \{\Omega_{n}(x; n)\}_{n \geq 0} \), \( \deg \Omega_{n}(x; n) = s, \quad n \geq 0, \) and a SCN \( \{s_{\nu, \omega}\}_{\nu = n-s}, \quad n \geq 0, \) such that

\[
\sum_{\nu = n}^{n+s} \theta_{n,\nu} Q_{\nu}(x) = \sum_{\nu = n-s}^{n+s} \theta^{[\nu]}_{n,\nu} B_{\nu}(x)
\]

\[
= \Omega_{s}(x; n)B_{n}(x) + \sum_{\nu = n-s}^{n-1} c^{[\nu]}_{n,\nu} B_{\nu}(x), \quad n \geq 0,
\]

(2.1)
where

$$\theta_{n,i}^{[0]} = \min(n,i) + s \sum_{\nu = \max(n,i)}^{n+s} \theta_{n,\nu} \lambda_{\nu,i}, \quad n-s \leq i \leq n+s, \ n \geq 0,$$

$$\theta_{r,r-s}^{[0]} = \theta_{r,r} \lambda_{r,r-s} \neq 0,$$

$$\sum_{\nu = n}^{m+s} \theta_{n,\nu} \nu = b_{m}^{-1}(u, \Omega_{s}(x; n)B_{n}B_{m}) + \zeta_{m,n}^{[0]},$$

$$n-s \leq m \leq n-1, \ n \geq 0,$$

$$\sum_{\nu = m}^{n+s} \theta_{n,\nu} \lambda_{\nu,m} = b_{m}^{-1}(u, \Omega_{s}(x; n)B_{n}B_{m}),$$

$$n \leq m \leq n-1, \ n \geq 0.$$

Proof. Let \((\theta_{n,i})^{n+s}_{n} \geq 0, \) where \(\theta_{n,n+s} = 1, \) \(n \geq 0,\) and \(\theta_{i,r} \neq 0, \) be a SCN. From (1.8) – (1.9), with \(t = 0\) and \(s \geq 1, \) we get

$$\sum_{\nu = n}^{n+s} \theta_{n,\nu} \nu = \sum_{\nu = n}^{n+s} \nu \sum_{i = -n}^{n} \theta_{n,i} \lambda_{\nu,i},$$

$$\sum_{\nu = m}^{n+s} \theta_{n,\nu} \lambda_{\nu,m} = b_{m}^{-1}(u, \Omega_{s}(x; n)B_{n}B_{m}),$$

where, for each pair of integers \((i, \nu)\) such that \(n-s \leq i \leq n+s\) and \(n \leq \nu \leq n+s, \) we took

$$\chi_{\nu} = \begin{cases} 1, & \text{if } \nu - s \leq i \leq \nu, \\ 0, & \text{otherwise}. \end{cases}$$

The permutation of these sums yields

$$\sum_{\nu = n}^{n+s} \theta_{n,\nu} \nu = \sum_{i = -n}^{n} \theta_{n,i} \lambda_{i},$$

where

$$\theta_{n,i}^{[0]} = \sum_{\nu = \max(n,i)}^{n+s} \theta_{n,\nu} \lambda_{\nu,i},$$

$$n-s \leq i \leq n+s, \ n \geq 0,$$

$$\theta_{r,r-s}^{[0]} = \theta_{r,r} \lambda_{r,r-s} \neq 0.$$

Hence, (2.2) and (2.3) are valid.

Multiplying by \(B_{n}(x)\) and using the orthogonality of \(\{B_{n}\}_{n \geq 0},\)

$$\sum_{i = 0}^{n+s} \theta_{n,i}^{[0]} \nu = b_{m}^{-1}(u, \Omega_{s}(x; n)B_{n}B_{m}) + \zeta_{m,n}^{[0]},$$

In particular, for \(0 \leq m \leq n-s-1 \) and \(n \geq s+1, \) it follows that \(\zeta_{m,n}^{[0]} = 0. \) Hence, (2.1) holds. Moreover, for \(n-s \leq m \leq n-1 \) and \(n \geq s, \) we recover (2.4).

Finally, for \(n \leq m \leq n-s+1 \) and \(n \geq 0, \) we deduce (2.5).

**Proposition 2.2.** Assume \(\{B_{n}\}_{n \geq 0}\) is a MPS and \(\{Q_{n}\}_{n \geq 0}\) fulfills (1.8) – (1.9), with \(t \geq 1. \) For any SCN \(\theta_{n,i}^{[0]} = \sum_{i = 0}^{n+s} \theta_{n,i} \lambda_{i}, \) \(n \geq 0, \) where \(\theta_{n,n+s} = 1, \) \(n \geq 0\) and \(\theta_{r,t,r} \neq 0, \) there exist a unique MPS \(\{Q_{n,s}(x; n)\}_{n \geq 0}, \) where \(\deg Q_{n,s}(x; n) = s + t, \) \(n \geq 0,\) and a SCN \(\phi_{n,s}^{[0]} = \sum_{i = 0}^{n+s} \phi_{n,i} \lambda_{i}, \) such that for every integer \(n \geq 0\)

$$\phi_{n,i}^{[0]} = \sum_{i = 0}^{n+s} \theta_{n,i} \lambda_{i},$$

$$\sum_{i = 0}^{n+s} \theta_{n,i} \lambda_{i} = b_{m}^{-1}(u, \Omega_{s}(x; n)B_{n}B_{m}) + \zeta_{m,n}^{[0]},$$

$$n-s \leq m \leq n-1.$$
From (1.8) – (1.9), we have
\[ P_n(x) = \sum_{\nu=-t}^{n} b_{n,\nu} \bar{P}_\nu(x), \quad n \geq t, \]
where \( \bar{\lambda}_{n,\nu} = \lambda_{n-t,\nu}, \quad n-t-s \leq \nu \leq n, \quad n \geq t, \) and \( \bar{\lambda}_{t+r+r-s} \neq 0. \) Now, let \( (\theta_{n,\nu})_{\nu=-t}^{n} \), \( n \geq 0, \) where \( \theta_{n+n+s} = 1, \) \( n \geq 0, \) and \( \theta_{r+t+r} \neq 0, \) be a SCN. One has
\[ \phi(x) = \sum_{\nu=-t}^{n+s} \theta_{n,\nu} Q_{\nu}(x) = \sum_{\nu=-t}^{n+s} \theta_{n,\nu} P_{\nu+t}(x) \]
\[ = \sum_{\nu=-t}^{n+s} \bar{\theta}_{n,\nu} P_{\nu}(x), \quad n \geq 0, \quad (2.13) \]
where \( \bar{\theta}_{n,\nu} = \theta_{n,\nu-\nu}, \quad n \leq \nu \leq n+t+s, \quad n \geq 0. \) Obviously, \( (\bar{\theta}_{n,\nu})_{\nu=-t}^{n+s} \), \( n \geq 0, \) is a SCN such that
\[ \bar{\theta}_{n,n+s+t} = \theta_{n,n+s}, \quad n \geq 0, \quad \bar{\theta}_{r+t+r,t} = \theta_{r+t+r} \neq 0. \]
But from Lemma 2.1, there exist a unique MPS \( \{\Omega_{t+s}(x; n)\}_{n \geq 0} \) and a SCN \( (\zeta_{n,t}^{[t]})_{n \geq 0, t \geq 0}, \) \( n \geq 0, \) such that
\[ \sum_{\nu=-n}^{n+t+s} \bar{\theta}_{n,\nu} P_{\nu}(x) = \sum_{i=-n-t}^{n+t+s} \tilde{\theta}_{n,i}^{[t]} B_i(x) \]
\[ = \Omega_{t+s}(x; n) B_n(x) + \sum_{\nu=-n-t}^{n-1} \zeta_{n,\nu} B_{\nu}(x), \quad (2.14) \]
for every integer \( n \geq 0, \) where
\[ \tilde{\theta}_{n,i}^{[t]} = \sum_{\nu=\max(n,i)+t}^{\min(n,i)+t+s} \bar{\theta}_{n,\nu} \bar{\lambda}_{\nu,t}, \quad n-t-s \leq i \leq n+t+s, \]
\[ \sum_{\nu=-n}^{n+t+s} \bar{\theta}_{n,\nu} \bar{\lambda}_{\nu,m} = b_{m-1}(u, \Omega_{t+s}(x; n) B_n B_m) + \zeta_{n,m}, \]
\[ n-t-s \leq m \leq n-1, \]
\[ \sum_{\nu=-n}^{n+t+s} \bar{\theta}_{n,\nu} \bar{\lambda}_{\nu,m} = b_{m-1}(u, \Omega_{t+s}(x; n) B_n B_m), \]
\[ n \leq m \leq n+t+s-1. \]
Finally, by using (2.13), (2.14), and taking into account the expressions of \( \bar{\lambda}_{n,\nu} \) and \( \bar{\theta}_{n,\nu}, \) we find the desired results.

3. A matrix approach and main results

In this section, we will work under the assumptions of the Proposition 2.2 and we will give a matrix approach to our problem.

If \( \Omega_{t+s}(x; n) = \sum_{\nu=0}^{t+s} \nu v_{n,\nu} x^\nu, \) \( n \geq 0, \) where \( v_{n,t+s} = 1, \) then relation (2.10) reads
\[ \sum_{\nu=0}^{n+m+t} \lambda_{n-t,m}^\nu \nu v_{n,\nu} = \sum_{\nu=0}^{t+s-1} b_{n,m}^\nu v_{n,\nu} + \zeta_{n,m}^t + b_{n,m}^{s+t}, \]
\[ n-s-t \leq m \leq n-1, \]
or, alternatively,
\[ \sum_{j=1}^{m+s+t-n+1} \lambda_{j+n-t-1,m}^{j+n+t-1} = \sum_{j=1}^{t+s} b_{n,m}^{j-1} v_{n,j-1} + \zeta_{n,m}^j + b_{n,m}^{s+j}, \]
for every \( n-s-t \leq m \leq n-1. \)

Replacing \( m \) by \( i + n - s - t - 1, \) we get
\[ \sum_{j=1}^{i} k_{i,j}^n \Theta_{n,j} = \sum_{j=1}^{t+s} b_{n,i+n-s-t-1}^{j-1} v_{n,i+j+n-s-t-1} + \]
\[ b_{n,i+n-s-t-1}^{j+n+t-1}, \quad 1 \leq j \leq i, \]
\[ 0, \quad \text{otherwise}, \]
and \( t_{i,j} = b_{n,i+n-s-t-1}^{j-1} v_{n,i+j+n-s-t-1}, \)
\[ \Theta_{n,j} = \beta_{n,i+j+n-s-t-1}, \] and \( V_{n,j} = v_{n,j-1}. \)
So we can write it as
\[ K_n \Theta_n = T_n V_n + W_n + E_n, \quad n \geq 0, \quad (3.1) \]
where
\[ K_n = (k_{i,j}^n)_{1 \leq i,j \leq s+t}, \quad T_n = (t_{i,j}^n)_{1 \leq i,j \leq s+t}, \]
\[ \Theta_n = \left( \Theta_{n,1}, \Theta_{n,2}, \ldots, \Theta_{n,s+t} \right)^T, \]
\[ V_n = \left( V_{n,1}, V_{n,2}, \ldots, V_{n,s+t} \right)^T, \]
\[ W_n = \left( w_{n-1}, w_{n-1-1}, \ldots, w_{n-1} \right)^T, \]
and
\[ E_n = \left( e_{n+s+t-1}, e_{n+s+t-1}, \ldots, e_{n+s+t-1} \right)^T. \]
In the same way, using $\theta_{n,n+s} = 1$, (2.11) can be written as:

$$
\sum_{\nu=m}^{n+s+t-1} \lambda_{\nu-t,n} \theta_{n,\nu-t} = \sum_{\nu=0}^{t+s-1} b_{n,m}^{\nu} v_{n,\nu} + b_{n,m}^{s+t} - \lambda_{n+s,m},
$$

or, equivalently,

$$
\sum_{j=m-n+1}^{s+t} \lambda_{j+n-t-1,n} \theta_{n,j+n-t-1} = \sum_{j=1}^{t+s} b_{n,m}^{j-1} v_{n,j-1} + b_{n,m}^{s+t} - \lambda_{n+s,m},
$$

for every $n \leq m \leq n+s+t-1$. Replacing $m$ by $i+n-1$, we get

$$
\sum_{j=i+n-1}^{s+t} m_{i,j}^{n} \theta_{n,j} = \sum_{j=1}^{t+s} s_{i,j}^{n} V_{n,j} + b_{n,i+n-1}^{s+t} - \lambda_{n+s,i+n-1},
\leq i \leq s + t,
$$

where for $i, j = 1, 2, \ldots, s + t$,

$$
m_{i,j}^{n} = \begin{cases} \lambda_{j+n-t-1,i+n-1}, & 1 \leq i \leq j \\ 0, & \text{otherwise}, \end{cases}
$$

$$
s_{i,j}^{n} = \begin{cases} b_{i+1,n-1}^{j-1}, & 1 \leq i \leq j \\ 0, & \text{otherwise}. \end{cases}
$$

Thus, we can use the matrix representation

$$
M_{n} \Theta_{n} = S_{n} V_{n} + F_{n}, \ n \geq 0,
\tag{3.2}
$$

where

$$
M_{n} = (m_{i,j}^{n})_{1 \leq i,j \leq s+t}, \quad S_{n} = (s_{i,j}^{n})_{1 \leq i,j \leq s+t},
$$

and

$$
F_{n} = (b_{n,n+1}^{s+t} - \lambda_{n+s,n+1}, b_{n,n+1}^{s+t} - \lambda_{n+s+1,n+1},
\ldots, b_{n,s+t-1,n+t-1}^{s+t} - \lambda_{n+s,s+t-1}).
$$

Our data are $\Theta_{n}$, $E_{n}$, $F_{n}$, $M_{n}$, $S_{n}$, $T_{n}$, $K_{n}$ and our unknowns are $V_{n}$ and $W_{n}$.

From (3.2), we get

$$
V_{n} = S_{n}^{-1}(M_{n} \Theta_{n} - F_{n}). \tag{3.3}
$$

Thus, substituting in (3.1) we get $K_{n} \Theta_{n} - W_{n} - E_{n} = T_{n} S_{n}^{-1}(M_{n} \Theta_{n} - F_{n})$, i.e.

$$
W_{n} = (K_{n} - T_{n} S_{n}^{-1} M_{n} \Theta_{n} + T_{n} S_{n}^{-1} F_{n} - E_{n}.
$$

As a consequence, for every choice of $\Theta_{n}$, we get $W_{n}$. From (3.3), we deduce $V_{n}$.

On the other hand, there exists a one-to-one correspondence between the vectors $W_{n}$ and $\Theta_{n}$ if and only if the matrix of dimension $s + t$, $K_{n} - T_{n} S_{n}^{-1} M_{n}$, is nonsingular.

Under such a condition, there exists a unique choice for $\Theta_{n}$ such that $W_{n} = 0$. Thus, we get

$$
\Theta_{n} = (K_{n} - T_{n} S_{n}^{-1} M_{n})^{-1} E_{n} - T_{n} S_{n}^{-1} F_{n},
$$

and from (3.3), $V_{n} = S_{n}^{-1} \Theta_{n} - S_{n}^{-1} F_{n}$. Then,

$$
V_{n} = (K_{n} M_{n}^{-1} S_{n} - T_{n})^{-1} E_{n} - [(K_{n} M_{n}^{-1} S_{n} - T_{n})^{-1} T_{n} + I_{s+t}] S_{n}^{-1} F_{n},
$$

where $I_{s+t}$ is the unit matrix. Hence, the polynomial $\Omega_{s+t}(x; n)$ is explicitly given.

Let introduce

$$
\Delta_{n}(t, s) = \det(K_{n} - T_{n} S_{n}^{-1} M_{n}), \ n \geq 0.
$$

Thus, we have proved the following result

**Proposition 3.1.** Assume $\{B_{n}\}_{n \geq 0}$ is a MOPS and $\{Q_{n}\}_{n \geq 0}$ fulfills (1.8) – (1.9). For a fixed integer $p \geq t + 1$, the following statements are equivalent.

i) $\Delta_{n}(t, s) \neq 0, \ n \geq p$.

ii) There exist a unique SCN $(\theta_{n,\nu}^{s})_{\nu=0}^{\infty}$, $n \geq p$, with $\theta_{n,n+s}^{s} = 1, \ n \geq p,$ and $\theta_{s+t,r} \neq 0,$ if $p \leq r + t$, and a unique MPS $\{\Omega_{s+t}^{*}(x; n)\}_{n \geq p},$ $\deg \Omega_{s+t}^{*}(x; n) = s + t, \ n \geq p$, such that

$$
\Omega_{s+t}^{*}(x; n) B_{n}(x) = \phi(x) \sum_{\nu=n-t}^{n} \theta_{n,\nu}^{s} Q_{\nu}(x), \tag{3.4}
$$

for $n \geq p$.

Our main result is

**Theorem 3.2.** Let $\{B_{n}\}_{n \geq 0}$ be a MOPS and $\{Q_{n}\}_{n \geq 0}$ be the MPS satisfying (1.8) – (1.9). For each fixed integer $p \geq t + 1$, if we suppose that $\phi(x)$ and $B_{n}(x)$ are coprime for every $n \geq p$, then the following statements are equivalent.

i) $\Delta_{n}(t, s) \neq 0, \ n \geq p$.

ii) There exist a unique SCN $(\theta_{n,\nu}^{s})_{\nu=0}^{\infty}$, $n \geq p$, where $\theta_{n,n+s}^{s} = 1, \ n \geq p,$ and $\theta_{s+t,r} \neq 0$ if $p \leq r + t$, and a unique MPS $\{\Omega_{s+t}^{*}(x; n)\}_{n \geq p},$ $\deg \Omega_{s+t}^{*}(x; n) = s + t, \ n \geq p$, such that

$$
\Omega_{s}^{*}(x; n) B_{n}(x) = \sum_{\nu=n-t}^{n} \theta_{n,\nu}^{s} Q_{\nu}(x), \ n \geq p. \tag{3.5}
$$
Proof. Taking into account \( \phi(x) \) and \( B_n(x) \) are coprime for every \( n \geq p \), from (3.4) we deduce that \( \phi \) divides \( \Omega_{s+i}(x; n), n \geq p \). So, \( \Omega_{s+i}(x; n) = \phi(x)\Omega^*_s(x; n), \ n \geq p \). Hence, the desired result follows.

The orthogonal polynomial sequence \( \{B_n\}_{n \geq 0} \) and the polynomial sequence \( \{Q_n\}_{n \geq 0} \) can be related by a general finite-type relation (see [1]). It reads as follows

\[
F(Q_n, \ldots, Q_{n-l}) = G(B_n, \ldots, B_{n-s}),
\]

where \( F \) and \( G \) are fixed functions.

When \( F \) and \( G \) are linear functions, some situations dealing with the inverse problem have been analyzed in [1,2]. There, necessary and sufficient conditions in order to \( \{Q_n\}_{n \geq 0} \) be orthogonal are obtained.

This kind of linear relations reads as follows.

There exists \( (l, s, r) \in \mathbb{N}^3 \), with \( r \geq \tilde{s} = \max(l, s) \) such that

\[
\sum_{\nu=\max(n-s)}^{n} \zeta_{n, \nu} Q_{\nu}(x) = \sum_{\nu=\max(n-s)}^{n} \lambda_{n, \nu} B_{\nu}(x), \quad n \geq \tilde{s},
\]

(3.6)

with \( \zeta_{r, r-\nu} \lambda_{r, r-s} \neq 0 \). Here, \( \zeta_{n, n} = \lambda_{n, n} = 1, \ n \geq \tilde{s} \).

More recently, in [5], A. M. Delgado and F. Marcellán exhaustively describe all the set of pairs of quasi-definite (regular) linear functional such that their corresponding sequences of monic polynomials \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) are related by a differential expression

\[
P_n(x) + s_n P_{n-1}(x) = R^{[1]}_n(x) + t_n R^{[1]}_{n-1}(x), \quad n \geq 1,
\]

where \( t_n \neq 0 \), for every \( n \geq 1 \), and with the technical condition \( t_1 \neq s_1 \).

Notice that in general \( \{R^{[1]}_n\}_{n \geq 0} \) is not a MOPS.

In the same context of our contribution, we show that the corresponding inverse finite-type relation between two sequences satisfying (3.6) is possible under certain conditions.

Indeed, let consider the MPS \( \{C_n\}_{n \geq \tilde{s}} \) given by

\[
C_n(x) = \sum_{\nu=\max(n-s)}^{n} \lambda_{n, \nu} B_{\nu}(x), \quad n \geq \tilde{s}.
\]

With the finite-type relation between the sequences \( \{C_n\}_{n \geq \tilde{s}} \) and \( \{Q_n\}_{n \geq \tilde{s}} \), we can associate the determinants \( \Delta_{n}(0, s) \), \( n \geq \tilde{s} \). So, we have.

**Corollary 3.3.** Let \( \{B_n\}_{n \geq 0} \) be a MOPS and \( \{Q_n\}_{n \geq 0} \) be the MPS satisfying (3.6). For each fixed integer \( p \geq \max(s, l, 1) \), if \( \Delta_{n}(0, s) \neq 0, n \geq p \), then there exist a unique SCN \( \{C_{n, \nu}\}_{\nu=\max(n, s)}^{n} \), \( n \geq p \), where \( \zeta_{n, n+s} = 1, n \geq p \), and \( \zeta_{r, r-\nu} \neq 0 \) if \( p \leq r \), and a unique MPS \( \{Q_{n}(x; n)\}_{n \geq p} \), deg \( Q_{n}(x; n) = s, n \geq p, \) such that

\[
\Omega^*_s(x; n)B_n(x) = \sum_{\nu=\max(n-s)}^{n} \zeta_{n, \nu} Q_{\nu}(x), \quad n \geq p.
\]

(3.8)

**Proof.** From Theorem 3.2, with \( t = 0 \), there exists the corresponding inverse finite-type relation associated with the relation (3.7) if and only if \( \Delta_{n}(0, s) \neq 0, n \geq p \). Equivalently, there exist a unique SCN \( \{\theta_{n, \nu}\}_{\nu=\max(n, s)}^{n} \), \( n \geq p \), where \( \theta_{n, n+s} = 1, n \geq p \), and \( \theta_{r, r-\nu} \neq 0 \), if \( p \leq r \), and a unique MPS \( \{Q_{n}(x; n)\}_{n \geq p} \), deg \( Q_{n}(x; n) = s, n \geq p, \) such that

\[
\Omega^*_s(x; n)B_n(x) = \sum_{\nu=\max(n-s)}^{n} \theta_{n, \nu} Q_{\nu}(x), \quad n \geq p.
\]

(3.9)

But from (3.6) and (3.7), the above expression becomes

\[
\Omega^*_s(x; n)B_n(x) = \sum_{\nu=\max(n-s)}^{n} \theta_{n, \nu} \sum_{i=\max(n-s)}^{n} \chi_{i, \nu} \zeta_{i, \nu} Q_{\nu}(x), \quad n \geq p,
\]

where, for each pair of integers \( (i, \nu) \) such that \( n - l \leq i \leq n + s \) and \( n \leq \nu \leq n + s \), we took

\[
\chi_{i, \nu} = \begin{cases} 1, & \text{if } \nu - l \leq i \leq \nu, \\ 0, & \text{otherwise} \end{cases}
\]

The permutation inside these two sums yields

\[
\Omega^*_s(x; n)B_n(x) = \sum_{i=\max(n-s)}^{n} \zeta_{n, i} \zeta_{i, \nu} Q_{\nu}(x),
\]

where

\[
\zeta_{n, i} = \min_{\nu=\max(n, i)}^{n+s} \theta_{n, \nu} \zeta_{i, \nu},
\]

if \( n - l \leq i \leq n + s, \ n \geq p, \) and \( \zeta_{r, r-\nu} \neq 0, \)

if \( p \leq r \).

\( \square \)
4. The case: $(t, s) = (0, 1)$

Let $\{B_n\}_{n \geq 0}$ be a MOPS with respect to the linear functional $u$ and satisfying the three-term recurrence relation (1.5).

Consider the following finite-type relation between $\{B_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, with index $s = 1$, with respect to $\phi(x) = 1$,

$$Q_n(x) = B_n(x) + \lambda_{n,n-1} B_{n-1}(x), \quad n \geq 0,$$  

$$\exists r \geq 1, \quad \lambda_{r,r-1} \neq 0. \quad (4.1)$$

From Lemma 2.1, for every set of complex numbers, $\theta_{n,n}$, $n \geq 0$, with $\theta_{r,r} \neq 0$, there exists a unique MPS $\Omega_1^s(x;n)$, where $\Omega_1(x;n) = x + \varepsilon_{n,0}$, $n \geq 0$, and a unique set of complex numbers, $\zeta_{n,n-1}$, $n \geq 0$, such that

$$Q_{n+1}(x) + \theta_{n,n} Q_n(x) = \Omega_1(x;n) B_n(x) + \zeta_{n,n-1} B_{n-1}(x), \quad n \geq 0.$$  

where

$$\begin{cases}
\lambda_{n,n-1} \theta_{n,n} = \zeta_{n,n-1} + \gamma_n, \quad n \geq 1, \\
\theta_{n,n} - \varepsilon_{n,0} = -\lambda_{n+n,1} + \beta_n, \quad n \geq 0.
\end{cases} \quad (4.4)$$

The determinants associated with (4.1) – (4.2) are given by

$$\Delta_0(0,1) = 0, \quad \Delta_n(0,1) = \lambda_{n,n-1}, \quad n \geq 1.$$  

where $\Delta_r(0,1) = \lambda_{r,r-1} \neq 0$. As a consequence of Theorem 3.2, when $t = 0$ and $s = 1$, we have the following result

**Proposition 4.1.** Let $\{B_n\}_{n \geq 0}$ be a MOPS and $\{Q_n\}_{n \geq 0}$ be the MPS satisfying (4.1) – (4.2). For every fixed integer $p \geq 1$, the following statements are equivalent

i) $\lambda_{n,n-1} \neq 0, \quad n \geq p.$

ii) There exist a unique set of complex numbers $\theta_{n,n}^* \neq 0, \quad n \geq p$, and a unique MPS $\Omega_1^s(x;n)$, $\deg \Omega_1^s(x;n) = 1, \quad n \geq p$, such that

$$\Omega_1^s(x;n) B_n(x) = Q_{n+1}(x) + \theta_{n,n}^* Q_n(x), \quad n \geq p.$$  

where

$$\theta_{n,n}^* = \frac{\gamma_n}{\lambda_{n,n-1}}, \quad n \geq p,$$  

$$\Omega_1^s(x;n) = x + \varepsilon_{n,0}.$$  

Then the following statements are equivalent

$$v_{n,0}^* = \frac{\gamma_n}{\lambda_{n,n-1}} + \lambda_{n+1,n} - \beta_n, \quad n \geq p.$$  

**Example.** In order to illustrate the result of Proposition 4.1, we study the structure relation characterizing a semi-classical polynomial sequence, $\{B_n\}_{n \geq 0}$, orthogonal with respect to the linear functional $u$ solution of the functional equation

$$u' + \psi u = 0,$$  

where $\psi(x) = -ix^2 + 2x - i(\alpha - 1)$ and with regularity condition $\alpha \notin \bigcup_{n \geq 0} E_n$, where $E_0 = \{\alpha \in \mathbb{C} : F(\alpha) = 0\}, \quad F(\alpha) = \int_{-\infty}^{+\infty} e^{ix - i\alpha x^2} dx$, and for each integer $n \geq 1, \quad E_n = \{\alpha \in \mathbb{C} : \Xi_n(\alpha) = 0\}$. Here, $\Xi_n(\alpha)$ is the Hankel determinant associated with $u$. Notice that $u$ is a semi-classical linear functional of class one [10].

The recurrence coefficients $\beta_n$ and $\gamma_{n+1}$, $n \geq 0$, of the sequence $\{B_n\}_{n \geq 0}$ are determined by the system [10]:

$$\begin{cases}
\lambda_{n+1} + 1 = 2 - i(\beta_n + \beta_{n+1}), \quad n \geq 0, \\
(\gamma_{n+2} + \gamma_{n+1}) = \psi(\beta_{n+1}), \quad n \geq 0, \\
\gamma_n = -i\psi(\beta_0), \quad \beta_0 = -i F'(0). \quad (4.10)
\end{cases}$$

The sequence $\{B_n\}_{n \geq 0}$ is characterized by the following structure relation [10]:

$$B_n^1(x) = B_{n+1}(x) - i\frac{n+1}{\beta_{n+1}} B_{n-1}(x), \quad n \geq 1.$$  

Thus, taking into account $\lambda_{n,n-1} = -\frac{i\gamma_n \gamma_{n+1}}{n+1} \neq 0, \quad n \geq 1$, we deduce a strictly finite-type relation between the sequences $\{B_n\}_{n \geq 0}$ and $\{B_n^1\}_{n \geq 0}$ with index $s = 1$, with respect to $\phi(x) = 1$.

From Proposition 4.1, we get the following inverse relation, for $n \geq 1$,

$$(x + v_{n,0}^*) B_n(x) = B_{n+1}^1(x) + i\frac{n+1}{\gamma_{n+1}} B_0^1(x), \quad (4.12)$$

where $v_{n,0}^* = -i\frac{n+1}{\gamma_{n+1} \gamma_{n+2}} - \beta_n, \quad n \geq 1$. The sequence $\{B_n\}_{n \geq 0}$ could be characterized by a relation as (4.12). It is the aim of the following result.

**Proposition 4.2.** Let $\{B_n\}_{n \geq 0}$ be a MOPS satisfying (1.5). Then the following statements are equivalent.

i) There exists a set of non-zero complex numbers $\{\lambda_{n,n-1}\}_{n \geq 1}$ such that, for $n \geq 1$,

$$B_n^1(x) = B_{n+1}(x) + \lambda_{n,n-1} B_{n-1}(x). \quad (4.13)$$
ii) There exists a set of complex numbers \( \{ \varrho_n \}_{n \geq 0} \), with \( \varrho_n \neq 0 \), \( n \geq 1 \), and \( \varrho_0 = 0 \), such that for \( n \geq 0 \),

\[
(x + \frac{\gamma_{n+1}}{\varrho_{n+1}} + \varrho_n - \beta_n)B_n(x) = B_{n+1}^{[1]}(x) + \varrho_n B_n^{[1]}(x) .
\]

(4.14)

Proof. Assume that i) holds. From Proposition 4.1, we get

\[
(x + \frac{\gamma_{n+1}}{\varrho_{n+1}} + \varrho_n - \beta_n)B_n(x) = B_{n+1}^{[1]}(x) + \varrho_n B_n^{[1]}(x) , \quad n \geq 1 ,
\]

where \( \varrho_n = \gamma_n \lambda_{n-1}^{-1}, \ n \geq 1 \). For \( n = 1 \), in (4.13), we obtain \( \lambda_{1,0} = \frac{\beta_0 - \beta_1}{2} \). Then, \( \gamma_1 = \frac{\beta_0 - \beta_1}{\varrho_1} \).

\[
(x + \frac{\gamma_1}{\varrho_1} - \beta_0)B_0(x) = x - \frac{\beta_0 + \beta_1}{2} = B_1^{[1]}(x) + \varrho_0 B_0^{[1]}(x) ,
\]

i.e. \( \varrho_0 = 0 \). Thus, ii) holds. Conversely, let us take \( \lambda_{n,n-1} = \gamma_n, \ n \geq 1 \), and consider the MPS \( \{ A_n \}_{n \geq 0} \) defined by

\[
A_n(x) = B_n(x) + \lambda_{n,n-1}B_{n-1}(x) , \quad n \geq 1 .
\]

(4.15)

From Proposition 4.1, we get

\[
(x + v_{n,0}^*)B_n(x) = A_{n+1}(x) + \theta_{n,n}^*A_n(x) , \quad n \geq 1 ,
\]

where \( v_{n,0}^* = \frac{\gamma_{n+1}}{\varrho_{n+1}} + \varrho_n - \beta_n \), \( n \geq 1 \), and \( \theta_{n,n}^* = \frac{\gamma_n}{\lambda_{n,n-1}} = \varrho_n \), \( n \geq 1 \). From the assumption ii) and the previous relation, it follows that

\[
\sum_{n=0}^{n+2} \theta_{n,n}Q_n(x) = \Omega_2(x,n)B_n(x) + \zeta_{0,n}^{[0]}B_{n-1}(x) + \zeta_{0,n-2}^{[0]}B_{n-2}(x) , \quad n \geq 0 ,
\]

(5.3)

where

\[
\begin{aligned}
\lambda_{n+1,n+1} + \theta_{n,n+1} & = \beta_{n+1} + \gamma_n + \varrho_n, \quad n \geq 0 , \\
\lambda_{2,0} + \theta_{0,1} & = \gamma_1 + \beta_0 (\gamma_0 + \varrho_0), \\
\lambda_{n+2,n} + \theta_{n,n+1} & = \gamma_{n+1} + \gamma_n + \beta_0 (\gamma_0 + \varrho_0), \quad n \geq 1 , \\
\theta_{n+1,n+1} + \lambda_{n+1,n} + \theta_{n,n} & = \gamma_n (\beta_n + \gamma_{n-1} + \varrho_{n-1}), \quad n \geq 1 , \\
\theta_{n,n} + \lambda_{n,n-2} & = \gamma_n (\gamma_n + \zeta_{0,n-2}), \quad n \geq 2 .
\end{aligned}
\]

The determinants associated with (5.1) – (5.2) are

\[
\Delta_0(0,2) = \Delta_1(0,2) = 0 , \quad \Delta_n(0,2) = \lambda_{n,n-2}(\lambda_{n+1,n-1} - \gamma_n), \quad n \geq 2 .
\]

(5.5)

As a consequence of Theorem 3.2, where \( t = 0 \) and \( s = 2 \), we have the following result

\[
A_{n+1}(x) + \varrho_n A_n(x) = B_{n+1}^{[1]}(x) + \varrho_n B_n^{[1]}(x) , \quad n \geq 1 .
\]

Equivalently,

\[
A_n(x) - B_n^{[1]}(x) = (\prod_{\nu=0}^{n-1} \varrho_{\nu}) (A_1(x) - B_1^{[1]}(x)) = 0 , \quad n \geq 1 .
\]

But, from (4.15) for \( n = 1 \) we get \( A_1(x) = x - \beta_0 + \frac{\gamma_1}{\varrho_1} \).

From (4.14), with \( n = 0 \), we get \( B_1^{[1]}(x) = x - \beta_0 + \frac{\gamma_1}{\varrho_1} \).

Hence, \( A_n(x) = B_n^{[1]}(x) , \ n \geq 0 \). Thus according to (4.15), i) holds.

\[
5. \text{ The case } (t,s) = (0,2)
\]

Let \( \{ B_n \}_{n \geq 0} \) be a MOPS with respect to the linear functional \( u \) and satisfying (1.5). Consider the following finite-type relation between \( \{ B_n \}_{n \geq 0} \) and \( \{ Q_n \}_{n \geq 0} \), with index \( s = 2 \), with respect to \( \phi(x) = 1 \), for \( n \geq 0 \),

\[
Q_n(x) = B_n(x) + \lambda_{n,n-1}B_{n-1}(x) + \lambda_{n,n-2}B_{n-2}(x) , \quad n \geq 0 ,
\]

(5.1)

\[
\exists \gamma_r, r \neq 0 .
\]

(5.2)

From Lemma 2.1, for every system of complex numbers \( \{ \theta_{n,v} \}_{v=0}^{n+2} \), \( n \geq 0 \), where \( \theta_{n,n+2} = 1 \), \( n \geq 1 \) and \( \theta_{r,r} \neq 0 \), there exists a unique MPS \( \{ \Omega_2(x,n) \}_{n \geq 0} \), where \( \Omega_2(x,n) = x^2 + v_{n,1}x + v_{n,0} \), \( n \geq 0 \), and a unique system of complex numbers, \( \{ \zeta_{n,v}^{[0]} \}_{v=0}^{n-2} \), \( n \geq 0 \), such that

\[
\Delta_0(0,2) = \Delta_1(0,2) = 0 , \quad \Delta_n(0,2) = \lambda_{n,n-2}(\lambda_{n+1,n-1} - \gamma_n), \quad n \geq 2 .
\]

(5.5)

Proposition 5.1. Let \( \{ B_n \}_{n \geq 0} \) be a MOPS and \( \{ Q_n \}_{n \geq 0} \) be the MPS satisfying (5.1) – (5.2). For every fixed integer \( p \geq 2 \), the following statements are equivalent

\[
i) \lambda_{n,n-2}(\lambda_{n+1,n-1} - \gamma_n) \neq 0 , \ n \geq p .
\]
ii) There exist a unique SCN \((\theta^{*}_{n,n})_{n=0}^{n+2}\), \(n \geq p\), with \(\theta_{n,n+2} = 1\), \(n \geq p\), and \(\theta^{*}_{r,r} \neq 0\), if \(p \leq r\), and there exists a unique MPS \((\Omega^{*}_{r}(x;n))_{n \geq p}\), where \(\deg \Omega^{*}_{r}(x;n) = 2\), \(n \geq p\), such that, for \(n \geq p\),

\[
\Omega^{*}_{r}(x;n)B_{n}(x) = Q_{n+2}(x) + \theta^{*}_{n,n+1}Q_{n+1}(x) + \theta^{*}_{n,n}Q_{n}(x).
\]

(5.6)

We write

\[
\theta^{*}_{n,n+1} = \frac{[\lambda_{n,n-2}(\beta_{n-1} - \beta_{n+1} + \lambda_{n+2,n+1}) - \lambda_{n,n-1} \gamma_{n} - 1]}{\lambda_{n,n-2}(\lambda_{n+1,n-1} - \gamma_{n})},
\]

\[
\theta^{*}_{n,n} = \frac{\gamma_{n} \gamma_{n-1} - 1}{\lambda_{n,n-2}}.
\]

\(\Omega^{*}_{r}(x;n) = x^{2} + v^{*}_{n,1}x + v^{*}_{n,0}, \ n \geq p\), (5.7)

where

\[
v^{*}_{n,0} = \theta^{*}_{n,n} + (\lambda_{n+1,n} - \beta_{n})\theta^{*}_{n+1,n+1} - \gamma_{n+1} - \gamma_{n} + \lambda_{n+2,n} + \beta_{n}(\lambda_{n+1,n} - \lambda_{n+2,n+1}),
\]

\[
v^{*}_{n,1} = \theta^{*}_{n,n+1} - \beta_{n+1} - \beta_{n} + \lambda_{n+2,n+1}.
\]

Example. Let \(\{B_{n}\}_{n \geq 0}\) be the sequence of monic polynomials, orthogonal with respect to the linear functional \(u\) such that

\[
\langle u, p \rangle = \int_{-\infty}^{+\infty} p(x)e^{-x^{2}}\,dx.
\]

This sequence of polynomials was introduced by P. Nevai (see [15]) in the framework of the so-called Freud measures. These polynomials satisfy the three-term recurrence relation (1.5), with coefficients \(\beta_{n} = 0\), \(n \geq 0\), and where \(\gamma_{n+1}\), \(n \geq 0\), are given by a non-linear recurrence relation (see [3] and [15])

\[
n = 4\gamma_{n}(\gamma_{n+1} + \gamma_{n} + \gamma_{n-1}), \ n \geq 1,
\]

with \(\gamma_{0} = 0\) and \(\gamma_{1} = \Gamma(3/4)\Gamma(1/4)\).

The sequence \(\{B_{n}\}_{n \geq 0}\) satisfies the following structure relation (see [3])

\[
B^{[1]}_{n}(x) = B_{n}(x) + \lambda_{n,n-2}B_{n-2}(x), \ n \geq 2,
\]

(5.8)

where

\[
\lambda_{n,n-2} = \frac{4}{n+1} \gamma_{n+1} \gamma_{n} - 1 \neq 0, \ n \geq 2.
\]

From (5.3), with \(Q_{n}(x) = B^{[1]}_{n}(x), \ n \geq 0\), and the fact that the polynomial sequences \(\{B_{n}\}_{n \geq 0}\) and \(\{B^{[1]}_{n}\}_{n \geq 0}\) are symmetric, i.e., \(B_{n}(-x) = (-1)^{n}B_{n}(x), \ n \geq 0\), we get, for \(n \geq 0\),

\[
B^{[1]}_{n+2}(x) + \theta^{*}_{n,n}B^{[1]}_{n}(x) = (x^{2} + v_{n,0})B_{n}(x) + \lambda^{[0]}_{n,n-2}B_{n-2}(x),
\]

(5.9)

where

\[
\begin{align*}
\lambda_{2,0} + \theta_{0,0} & = \gamma_{1} + v_{0,0}, \\
\lambda_{n+2,n} + \theta_{n,n} & = \gamma_{n+1} + v_{n,0}, \ n \geq 1, \\
\theta_{n,n}\lambda_{n,n-2} & = \gamma_{n} - v_{n,n-2}, \ n \geq 2.
\end{align*}
\]

Since we have \(\lambda_{n,n-2}\), \(n \geq 2\), the choice \(v_{n,n-2} = 0\), \(n \geq 2\), is possible and yields the inverse relation

\[
(x^{2} + v_{n,0})B_{n}(x) = B^{[1]}_{n+2}(x) + \theta^{*}_{n,n}B^{[1]}_{n}(x), \ n \geq 0,\ (5.11)
\]

where

\[
\theta^{*}_{n,n} = \frac{n+1}{4\gamma_{n+1}},
\]

\[
v^{*}_{n,0} = \frac{n+1}{4\gamma_{n+1}} - \gamma_{n} + \gamma_{n+1} + 4 + 3\gamma_{n+1} + 2\gamma_{n+3}.
\]

Here, the determinants associated with (5.8) are

\[
\Delta_{n}(0,2) = \frac{4}{n+1} \gamma_{n+1} \gamma_{n} - 1 \left[\frac{4}{n+2} + 2\gamma_{n+1} - 1\right],
\]

(5.12)

where \(n \geq 2\), with \(\Delta_{0}(0,2) = \Delta_{1}(0,2) = 0\).

From Proposition 5.1, we deduce that the uniqueness of the previous inverse relation requires that \(\lambda_{n+1,n-1} = \gamma_{n} = \gamma_{n+1} = \frac{4}{n+2} + 2\gamma_{n+1} - 1 \neq 0, \ n \geq 2\). Equivalently, \(4\gamma_{n+1} + 2\gamma_{n+1} - 1 \neq n \geq 2\). Indeed, by using (5.8), where \(v\) is replaced by \(n + 1\) and taking into account the orthogonality of the polynomial sequence \(\{B_{n}\}_{n \geq 0}\), we get \(B^{[1]}_{n+1}(x) = xB_{n}(x) + (\lambda_{n+1,n-1} - \gamma_{n})B_{n-1}(x), \ n \geq 1\). On the other hand, if we suppose that there exists an integer \(n_{0} \geq 2\) such that \(\lambda_{n+1,n-1} - \gamma_{n} = 0\), then \(B^{[1]}_{n+1}(x) = xB_{n}(x)\). In this case (5.11), with \(n = n_{0}\) will be written as \((x^{2} + \alpha x + v_{n,0})B_{n}(x) = B^{[1]}_{n+2}(x) + \alpha B^{[1]}_{n+1}(x) + \theta^{*}_{n,n}B^{[1]}_{n}(x), \ n \geq 0\), and \(\alpha \in \mathbb{C}\). This contradicts the uniqueness of the inverse relation.

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